Calculus on fractal subsets of real line – I: formulation

Abhay Parvate†‡ and A D Gangal†

† Department of Physics, University of Pune, Pune 411 007, India
‡ Centre for Modeling and Simulation, University of Pune, Pune 411 007, India

Abstract. A new calculus based on fractal subsets of the real line is formulated. In this calculus, an integral of order $\alpha, 0 < \alpha \leq 1$, called F^α-integral, is defined, which is suitable to integrate functions with fractal support F of dimension α. Further, a derivative of order $\alpha, 0 < \alpha \leq 1$, called F^α-derivative, is defined, which enables us to differentiate functions, like the Cantor staircase, "changing" only on a fractal set. The F^α-derivative is local unlike the classical fractional derivative. The F^α-calculus retains much of the simplicity of ordinary calculus. Several results including analogues of fundamental theorems of calculus are proved.

The integral staircase function, which is a generalisation of the functions like the Cantor staircase function, plays a key role in this formulation. Further, it gives rise to a new definition of dimension, the γ-dimension.

Spaces of F^α-differentiable and F^α-integrable functions are analyzed. Analogues of Sobolev Spaces are constructed on F and F^α-differentiability is generalized using Sobolev-like construction.

F^α-differential equations are equations involving F^α-derivatives. They can be used to model sublinear dynamical systems and fractal time processes, since sublinear behaviours are associated with staircase-like functions which occur naturally as their solutions. As examples, we discuss a fractal-time diffusion equation, and one dimensional motion of a particle undergoing friction in a fractal medium.

Keywords: Cantor Functions, Dimensions, Fractal Integral, Fractal Derivative, Fractal Differential Equations, Sobolev Spaces on Sublinear Fractals

E-mail: `abhay@physics.unipune.ernet.in`, `adg@physics.unipune.ernet.in`

1. Introduction

It is now a well established fact that fractals can model many structures found in nature [1, 2]. The geometry of fractals is also a well explored subject [1, 3, 4, 5, 6].

Fractals are often too irregular to have any smooth differentiable structure defined on them, and render the methods and techniques of ordinary calculus powerless or inapplicable. For example the derivative of a Lebesgue-Cantor staircase function is zero almost everywhere and therefore this function is not a solution of an ordinary differential equation. Consequently, ordinary calculus does not equip us to handle problems such as fractal time random walks, anomalous diffusion, dynamics on fractals, fields of fractally distributed sources etc., by setting up and solving ordinary differential equations.

During recent times, a few approaches have been developed to deal with various aspects of the problems mentioned above.

Brownian motion and diffusion are very mature subjects [7]. However, several authors have recognized the need to use fractional derivatives and integrals to explore the characteristic features of fractal walks, anomalous diffusion, transport, etc. by setting up fractional kinetic equations, master equations and so on [8, 9, 10, 11, 12, 13, 14, 15, 16]. Fractional derivatives are nonlocal operators and often are suitable for modelling processes with memory but not always suitable to handle the local scaling behaviour e. g. the behaviour of fractal functions. In [17, 18, 19, 20] this problem was circumvented by renormalising fractional derivatives and constructing local fractional operators. This was further pursued in [21, 22]. A particular success of this approach was the demonstration of the striking fact that fractal and multifractal functions can be differentiated upto an order (fractional) determined by the Holder exponent of the function (or dimension of its graph). In particular, Weierstrass' nowhere differentiable function was shown [17, 18] to be differentiable upto order $(1-\gamma)$, if $(1+\gamma)$ is the box dimension of its graph. Similar results were obtained using conventional operators in [23].

Another remarkable development is analysis on (connected) fractals. Many important ideas and applications are developed in the realm of analysis on fractals. This approach has been extensively used for the treatment of diffusion, heat conduction, waves, etc. on fractals—see [24, 25, 26, 27, 28] and several references therein.

There is a further beautiful developement using measure-theoretical approaches, mainly related to disconnected fractal subsets of $\mathbf{R}$. It includes themes such as defining derivative as the inverse of the integral with respect to a measure and defining other operators using the derivative [29, 30], further generalizations [31, 32], and exploring relation between similarity dimension and spectral dimension [33].

While all these themes have increased our understanding and brought out many beautiful connections, a direct and simple approach involving fractional order operators on fractal sets is only moderately explored. Even though measure theoretical approach is elegant, Riemann integration like procedures have their own place. They are more transparent, constructive, and advantageous from algorithmic point of view. Rarely numerical algorithms are based on Lebesgue integrals but are based more often on Riemann sums. For example, numerical integration algorithms, varying from general ones such as Simpson and Gaussian quadratures and Runge-Kutta methods, to specialized ones such as symplectic Runge-Kutta methods are based on Riemann integral.

It indeed seems possible to develope such an appropriate calculus, tuned to these requirements. In the present paper, the first of a series devoted to these ideas, we undertake a systematic developement of calculus on fractal subsets of real line, involving integrals and derivatives of appropriate orders $\alpha \in (0, 1]$. A brief glance at the table in Appendix C would reveal that much of the simplicity and intuitive appeal of ordinary calculus can be retained. Further it is possible to identify the relevent functions spaces. In particular one can define Soboleve spaces on fractal subsets of the real line.

Differential equations of the form $D^\alpha f(x) = g(x)$, where D^α is a local differential operator of order α, $0 < \alpha < 1$ were considered in [19]. It was argued that such equations can have meaningful solutions if the dimension of the support of g is α. Then the fractional integral of the characteristic function of a Cantor set should be the corresponding Cantor staircase function, while the local fractional derivative of the latter should be the former. This was done using a Riemann integral like prescription. In particular a local fractional diffusion equation was shown to have subdiffusive solutions. This enabled one to identify a new exact solution of Chapman-Kolmogorov equation.

This paper formulates the ideas of integral and derivative of order α, $0 < \alpha \leq 1$ based on a (fractal) set $F \subset \mathbf{R}$, indicated in [19]. We call them F^α-integral and F^α-derivative respectively.

The organisation of the paper is as follows: We begin in section 2 by defining a mass function $\gamma^\alpha(F, a, b)$ and integral staircase function S_F^α of an order α for a set F. The mass function $\gamma^\alpha(F, a, b)$ gives us the content of a set F in the interval $[a, b] \subset \mathbf{R}$. Its definition is based on Riemann-like sums. The construction can be compared to the definition of Hausdorff measure, except that the covers are more restrictive: they are in the form of finite subdivisions of $[a, b]$. Though γ^α is not a measure due to this simplification, it turns out to be proportional to Hausdorff measure for compact sets. The integral staircase function $S_F^\alpha(x) = \gamma^\alpha(F, a, x)$, obtained from the mass function by fixing a, is a generalization of the well known functions such as the Lebesgue-Cantor staircase (or the Devil's staircase) functions. The definitions of F^α-integral and F^α-derivative use the quantity $(S_F^\alpha(y) - S_F^\alpha(x))$ in place of the length $(y - x)$ of the interval $[x, y]$. In this respect, the definition of F^α-integral is similar to that of Riemann-Stieltjes integral [34, 35].

In section 3 it is shown that the mass function leads to a new definition of dimension called γ-dimension, which is finer than the box dimension, though not as fine as the Hausdorff dimension. In later sections it is seen that the F^α-integral or the F^α-derivative give meaningful results when the γ-dimension of the underlying fractal F is the same as α.

Section 4 compares the mass function and the γ-dimension with standard definitions: mainly the box dimension and the Hausdorff measure and dimension. It is shown that the γ-dimension is a quantity in between the box and the Hausdorff dimension, and cases of equality and inquality with both have been illustrated. This includes the theorem stating that for compact sets, the Hausdorff measure and the mass function are proportional and the dimensions equal.

In section 5 the algorithmic nature of the mass function and the γ-dimension is discussed. A couple of results due to a preliminary algorithm are presented. The actual algorithm is given in Appendix D.

Several sets can give rise to the same staircase function. A representative set from such an equivalence class of sets, with nice properties, needs to be chosen for defining the F^α-derivative and proving the analogues of fundamental theorems. Section 6 assures the existence and uniqueness of such a set, called an α-perfect set, associated with a staircase function.

We develope the rest of the theory in a way analogous to the standard calculus [34, 35, 36]. In section 7 we introduce notations for limit and continuity using the topology of F with the metric inherited from $\mathbf{R}$. This is done in order to distinguish them from limit and continuity on $\mathbf{R}$.

The ordinary integral of functions with fractal support $F \subset \mathbf{R}$ is zero or undefined depending on the definition of integral (Lebesgue or Riemann) and the nature of the support. The F^α-integral defined in section 8 suits the needs of integration of such functions. It is further shown that the F^α-integral of the characteristic function of F is the staircase function associated with F as indicated in [19]. The similarity and differences between F^α-integral and Riemann-Stieltjes integral are pointed out.

Functions representing fractal time evolution typically “change” only on a fractal. The Cantor staircase function is an example. The F^α-derivative defined in section 9 is best suited to quantify the “rate of change” of such functions. This derivative is local unlike fractional derivatives [37, 38, 39, 40]. It is also not any kind of average derivative as in [41, 42, 43, 44, 45]. It is more like the first order derivative in ordinary calculus, which makes its dynamical interpretation possible. Further it is a constructive definition allowing direct algorithmic implementation (unlike one in the measure-theoretical approach). As an important characteristic, it is shown that the F^α-derivative of the staircase function of F is the characteristic function of F. Analogues of Rolle’s theorem, the law of the mean and Leibniz rule are discussed. In section 10, we prove the analogues of fundamental theorems of calculus. The formula for F^α-integration by parts follows thereby.

The definitions of F^α-integral and the F^α-derivative reduce to those of usual Riemann integral and first order derivative respectively when $\alpha = 1$ and $F = \mathbf{R}$.

In section 11, relevent function spaces are discussed. In particular, the spaces of F^α-differentiable functions and F^α-integrable functions are explored. F^α-differentiability is generalized using Sobolev-like construction. Completeness and separability of these spaces have been proved.

In section 12 we discuss a couple of examples demonstrating the use of F^α-differential equations. The first of them is a model for subdiffusion, which we contrast with one using fractional derivatives [46]. The second is a model for motion of a particle undergoing friction in a fractal medium. We further illustrate, in another example, the generality offered by a differential equation in an appropriate Sobolev space.

As an example of calculating F^α-integrals analytically, the F^α-integral of $f(x) = x\,\chi_C(x)$ for the middle $\frac{1}{3}$ Cantor set C is calculated in Appendix A. Repeated F^α-derivatives and F^α-integrals are discussed in Appendix B, where we also calculate F^α-derivatives and F^α-integrals of powers $(S_F^\alpha(x))^n$. A few analogies between classical calculus and F^α-calculus are tabulated in Appendix C. Appendix D is devoted to an algorithm for computing the mass function and the γ-dimension.

We begin by defining the integral staircase function.

2. The mass function and the integral staircase

Let F be a subset of the real line. In most of the interesting cases discussed below, F would be a fractal. In this section we formulate the notion of the content or *α-mass* of F in an interval $[a,b]$, i. e. mass of $F \cap [a,b]$, of order $\alpha, 0 < \alpha \leq 1$.

In all the following discussion, $0 < \alpha \leq 1$ unless stated otherwise.

Definition 1 *The* flag function *$\theta(F,I)$ for a set F and a closed interval I is given by*

$$\theta(F,I) = \begin{cases} 1 & \text{if } F \cap I \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Definition 2 *A* subdivision *$P_{[a,b]}$ (or just P) of the interval $[a,b]$, $a < b$, is a finite set of points $\{a = x_0, x_1, \cdots, x_n = b\}$, $x_i < x_{i+1}$. Any interval of the form $[x_i, x_{i+1}]$ is called a* component interval *or just a* component *of the subdivision P. If Q is any subdivision of $[a,b]$ and $P \subset Q$, then we say that Q is a* refinement *of P. If $a = b$, then the set $\{a\}$ is the only subdivision of $[a,b]$.*

Definition 3 *For a set F and a subdivision $P_{[a,b]}$, $a < b$,*

$$\sigma^\alpha[F,P] = \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha+1)} \theta(F,[x_i,x_{i+1}]). \tag{2}$$

If $a = b$, we define $\sigma^\alpha[F,P]$ to be zero.

We note that the sum in (2) contains a contribution from a component interval if and only if that component contains at least one point of F. Further, $\sigma^\alpha[F,P] \geq 0$ for any set F and subdivision P of $[a,b]$.

We remark that this definition, and in particular the factor $1/\Gamma(\alpha+1)$ and the use of finite subdivisions, has been motivated by local fractional calculus [19, 20].

Now we introduce the coarse-grained mass:

Definition 4 *Given $\delta > 0$ and $a \leq b$, the coarse-grained mass $\gamma_\delta^\alpha(F,a,b)$ of $F \cap [a,b]$ is given by*

$$\gamma_\delta^\alpha(F,a,b) = \inf_{\{P_{[a,b]}:|P|\leq\delta\}} \sigma^\alpha[F,P] \tag{3}$$

where

$$|P| = \max_{0 \le i \le n-1} (x_{i+1} - x_i) \tag{4}$$

for a subdivision P, and the infimum in (3) is taken over all subdivisions P of $[a, b]$ satisfying $|P| \le \delta$.

Eventually, a limit of $\gamma_\delta^\alpha(F, a, b)$ as $\delta \to 0$ is taken in definition 8 below. But before that we examine some important properties of $\gamma_\delta^\alpha(F, a, b)$.

Let $a \le b$ and $0 < \delta_1 < \delta_2$. Then $\gamma_{\delta_1}^\alpha(F, a, b)$ is the infimum of $\sigma^\alpha[F, P]$ over a smaller class of subdivisions than $\gamma_{\delta_2}^\alpha(F, a, b)$. Thus:

Lemma 5 *Let $a \le b$ and $0 < \delta_1 < \delta_2$. Then $\gamma_{\delta_1}^\alpha(F, a, b) \ge \gamma_{\delta_2}^\alpha(F, a, b)$.*

The following lemma shows that $\gamma_\delta^\alpha(F, a, b)$ is a monotonic increasing function of b and a monotonic decreasing function of a. (Throughout the paper we distinguish between *monotonic* and *strictly monotonic.*)

Lemma 6 *Let $\delta > 0$ and $a < b < c$. Then, $\gamma_\delta^\alpha(F, a, b) \le \gamma_\delta^\alpha(F, a, c)$ and $\gamma_\delta^\alpha(F, b, c) \le \gamma_\delta^\alpha(F, a, c)$.*

Proof: Let $\epsilon > 0$. Then according to the definition 4 of $\gamma_\delta^\alpha(F, a, c)$, there exists a subdivision $P_{[a,c]} = \{x_0 = a, x_1, \ldots, x_n = c\}$ such that $|P| \le \delta$ and

$$\sigma^\alpha[F, P] < \gamma_\delta^\alpha(F, a, c) + \epsilon.$$

Let $Q_{[a,b]} = \{x \in P : x < b\} \cup \{b\}$ i. e. $Q_{[a,b]} = \{y_0, y_1, \ldots, y_m\}$ where $y_i = x_i$ if $x_i < b$ and $y_m = b$.

It follows that $|Q_{[a,b]}| \le |P_{[a,c]}| \le \delta$ and $\theta(F, [y_{m-1}, y_m]) \le \theta(F, [x_{m-1}, x_m])$ since $[y_{m-1}, y_m] \subset [x_{m-1}, x_m]$. Therefore,

$$\sigma^\alpha[F, Q_{[a,b]}] \le \sigma^\alpha[F, P_{[a,c]}] < \gamma_\delta^\alpha(F, a, c) + \epsilon.$$

But $\gamma_\delta^\alpha(F, a, b) \le \sigma^\alpha[F, Q]$ and ϵ is arbitrary, hence

$$\gamma_\delta^\alpha(F, a, b) \le \gamma_\delta^\alpha(F, a, c).$$

which completes the proof of the first part. The second part follows in a similar way. •

Theorem 7 *$\gamma_\delta^\alpha(F, a, b)$ is continuous in b and a.*

Proof: We prove the continuity of $\gamma_\delta^\alpha(F, a, b)$ in b (with δ, α and a fixed). Continuity in a can be proved in a similar manner.

Given $\epsilon > 0$, let

$$\Delta' = (\epsilon\, \Gamma(\alpha + 1))^{\frac{1}{\alpha}} \text{ and } \Delta = \min(\Delta', \delta).$$

For $\epsilon_1 > 0$, there exists a subdivision P, such that $|P| \le \delta$ and

$$\sigma^\alpha[F, P] < \gamma_\delta^\alpha(F, a, b) + \epsilon_1.$$

Now $Q = P \cup \{b + \Delta\}$ is a subdivision of $[a, b + \Delta]$. Therefore,

$$\begin{aligned}
\gamma_\delta^\alpha(F, a, b + \Delta) &\leq \sigma^\alpha[F, Q] \\
&= \sigma^\alpha[F, P] + \theta(F, [b, b + \Delta])\frac{\Delta^\alpha}{\Gamma(\alpha + 1)} \\
&\leq \sigma^\alpha[F, P] + \epsilon \\
&< \gamma_\delta^\alpha(F, a, b) + \epsilon_1 + \epsilon.
\end{aligned}$$

As ϵ_1 is arbitrary, we get $\gamma_\delta^\alpha(F, a, b+\Delta) < \gamma_\delta^\alpha(F, a, b)+\epsilon$. As $\gamma_\delta^\alpha(F, a, b)$ is a nondecreasing function of b,

$$\gamma_\delta^\alpha(F, a, b + t) < \gamma_\delta^\alpha(F, a, b) + \epsilon$$

for $0 < t < \Delta$.

Summarizing, given $\epsilon > 0$, there exists a $\Delta > 0$ such that

$$c - b < \Delta \Longrightarrow \gamma_\delta^\alpha(F, a, c) - \gamma_\delta^\alpha(F, a, b) < \epsilon$$

which implies that $\gamma_\delta^\alpha(F, a, b)$ is continuous in b from *right*. The continuity from left follows on the replacement of b by $b - \Delta$ and of $b + \Delta$ by b in the above proof. •

As mentioned earlier, the mass function is the limit of the coarse-grained mass as $\delta \to 0$:

Definition 8 *The mass function $\gamma^\alpha(F, a, b)$ is given by*

$$\gamma^\alpha(F, a, b) = \lim_{\delta \to 0} \gamma_\delta^\alpha(F, a, b).$$

We note that since $\gamma_\delta^\alpha(F, a, b)$ increases as δ decreases, $\gamma^\alpha(F, a, b)$ always exists and is a non-negative number, which may possibly be $+\infty$.

Another simple observation is that if $F \cap [a, b] = \emptyset$, then $\gamma_\delta^\alpha(F, a, b) = 0$ for any $\delta > 0$, and consequently $\gamma^\alpha(F, a, b) = 0$. This result can be extended so that it also applies to an open interval (a, b):

Lemma 9 *If $F \cap (a, b) = \emptyset$, then $\gamma^\alpha(F, a, b) = 0$.*

Proof: If $F \cap [a, b] = \emptyset$, then the result is obvious. If not, then $F \cap [a, b]$ contains one or both of a and b. In that case, given $\epsilon > 0$, we can choose a subdivision P of $[a, b]$ such that

$$(x_1 - x_0) \text{ and } (x_n - x_{n-1}) < \left(\frac{\epsilon\,\Gamma(\alpha + 1)}{2}\right)^{\frac{1}{\alpha}}$$

where $\{x_0, \ldots, x_n\}$ are points of P. Then, $\sigma^\alpha[F, P] < \epsilon$ since $[x_1, x_{n-1}] \cap F = \emptyset$. But as ϵ is arbitrary, it follows that $\gamma_\delta^\alpha(F, a, b) = 0$ for any $\delta > 0$, so that $\gamma^\alpha(F, a, b) = 0$. •

A property, desired of a mass function, is additivity. The following theorem asserts this.

Theorem 10 *Let $a < b < c$ and $\gamma^\alpha(F, a, c) < \infty$. Then*

$$\gamma^\alpha(F, a, c) = \gamma^\alpha(F, a, b) + \gamma^\alpha(F, b, c). \tag{5}$$

Proof: Given $\delta > 0$, let P_1 be any subdivision of $[a, b]$ and P_2 be any subdivision of $[b, c]$ such that $|P_1| \le \delta$ and $|P_2| \le \delta$. Then, $P_1 \cup P_2$ is a subdivision of $[a, c]$, $|P_1 \cup P_2| \le \delta$, and

$$\sigma^\alpha[F, P_1 \cup P_2] = \sigma^\alpha[F, P_1] + \sigma^\alpha[F, P_2].$$

Taking infimum over all subdivisions P_1 and P_2 such that $|P_1| \le \delta$ and $|P_2| \le \delta$, and noting that *not* all the subdivisions of $[a, c]$ can be written in the form $P_1 \cup P_2$, where P_1 is a subdivision of $[a, b]$ and P_2 is that of $[b, c]$, we get

$$\begin{aligned}\gamma_\delta^\alpha(F, a, c) &\le \inf_{|P_1|\le\delta, |P_2|\le\delta} \sigma^\alpha[F, P_1 \cup P_2] \\ &= \gamma_\delta^\alpha(F, a, b) + \gamma_\delta^\alpha(F, b, c). \end{aligned} \tag{6}$$

Now for every subdivision $P_{[a,c]}$, $|P| \le \delta$, we can construct a subdivision $P' = P \cup \{b\}$. Obviously $|P'| \le \delta$, and $P' = P_1 \cup P_2$ where P_1 is a subdivision of $[a, b]$ and P_2 is a subdivision of $[b, c]$.

Let $P = \{x_0, x_1, \ldots, x_n\}$. If $b \in P$, then $P = P'$ and $\sigma^\alpha[F, P] = \sigma^\alpha[F, P']$. Otherwise, let $[x_k, x_{k+1}]$ be the interval which contains b. Thus,

$$\begin{aligned}\sigma^\alpha[F, P \cup \{b\}] - \sigma^\alpha[F, P] = \theta(F, [x_k, b])\frac{(b - x_k)^\alpha}{\Gamma(\alpha + 1)} + \theta(F, [b, x_{k+1}])\frac{(x_{k+1} - b)^\alpha}{\Gamma(\alpha + 1)} \\ - \theta(F, [x_k, x_{k+1}])\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha + 1)}.\end{aligned}$$

Hence,

$$\sigma^\alpha[F, P \cup \{b\}] - \sigma^\alpha[F, P] \le \frac{3\delta^\alpha}{\Gamma(\alpha + 1)}.$$

This implies that

$$\begin{aligned}\sigma^\alpha[F, P] + \frac{3\delta^\alpha}{\Gamma(\alpha + 1)} &\ge \sigma^\alpha[F, P \cup \{b\}] \\ &= \sigma^\alpha[F, P_1] + \sigma^\alpha[F, P_2] \\ &\ge \gamma_\delta^\alpha(F, a, b) + \gamma_\delta^\alpha(F, b, c)\end{aligned}$$

for all P. Thus if we take infimum over all subdivisions P such that $|P| \le \delta$, we get

$$\gamma_\delta^\alpha(F, a, c) + \frac{3\delta^\alpha}{\Gamma(\alpha + 1)} \ge \gamma_\delta^\alpha(F, a, b) + \gamma_\delta^\alpha(F, b, c) \tag{7}$$

From (6) and (7) and taking the limit as $\delta \to 0$, we get the result. •

Since each term in (5) is nonnegative for $a \le b \le c$, an immediate consequence is

Corollary 11 *$\gamma^\alpha(F, a, b)$ is increasing in b and decreasing in a.*

The next theorem states that $\gamma^\alpha(F, a, x)$ takes all values in the range $(0, \gamma^\alpha(F, a, b))$ for $x \in (a, b)$.

Theorem 12 *Let $a < b$ and let $\gamma^\alpha(F, a, b) \ne 0$ be finite. Let y be such that $0 < y < \gamma^\alpha(F, a, b)$. Then there exists c, $a < c < b$, such that $\gamma^\alpha(F, a, c) = y$.*

Proof: Let $z = \gamma^\alpha(F, a, b) - y$.

Given a $\delta > 0$, consider the set of all points x of $[a, b]$ such that $\gamma_\delta^\alpha(F, x, b) \leq z$. This set is an interval of the form $[s_\delta, b]$ for some s_δ, $a \leq s_\delta < b$, because $\gamma_\delta^\alpha(F, x, b)$ is continuous (theorem 7) and decreasing in x (corollary 11). Since $\gamma_\delta^\alpha(F, x, b)$ increases as δ decreases (lemma 5), s_δ increases as δ decreases.

Similarly the set of all points x of $[a, b]$ such that $\gamma_\delta^\alpha(F, a, x) \leq y$ is an interval of the form $[a, t_\delta]$, $a < t_\delta \leq b$, and t_δ decreases as δ decreases.

Let $x \in (a, b)$. Then by theorem 10,

$$\gamma^\alpha(F, a, b) = \gamma^\alpha(F, a, x) + \gamma^\alpha(F, x, b) \geq \gamma_\delta^\alpha(F, a, x) + \gamma_\delta^\alpha(F, x, b). \tag{8}$$

As $y, z < \gamma^\alpha(F, a, b)$, there exists a $\delta_0 > 0$ such that $\delta < \delta_0$ implies that $\gamma_\delta^\alpha(F, a, b) > y, z$. In the rest of this proof, we only consider $\delta < \delta_0$ without mentioning.

Since $\gamma_\delta^\alpha(F, a, b) > y$ and $\gamma_\delta^\alpha(F, a, u)$ is continuous and increasing in u, there exists an $x \in (a, b)$ such that $\gamma_\delta^\alpha(F, a, x) = y$. This implies that $x \in [a, t_\delta]$. Further, from (8), it follows that

$$z = \gamma^\alpha(F, a, b) - y = \gamma^\alpha(F, a, b) - \gamma_\delta^\alpha(F, a, x) \geq \gamma_\delta^\alpha(F, x, b)$$

implying that x also belongs to $[s_\delta, b]$. This can happen only when $s_\delta \leq t_\delta$.

Thus for each δ there exists an interval $[s_\delta, t_\delta]$ such that

$$x \in [s_\delta, t_\delta] \Longrightarrow \gamma_\delta^\alpha(F, x, b) \leq z \text{ and } \gamma_\delta^\alpha(F, a, x) \leq y.$$

Let $s = \sup_{0<\delta<\delta_0} s_\delta$ and let $t = \inf_{0<\delta<\delta_0} t_\delta$. Now s_δ increases and t_δ decreases as δ goes to zero, but as $s_\delta \leq t_\delta$ for any δ. Thus $s \leq t$ and

$$[s, t] = \bigcap_{0<\delta<\delta_0} [s_\delta, t_\delta].$$

Consequently $x \in [s, t]$ implies $\gamma_\delta^\alpha(F, x, b) \leq z$ and $\gamma_\delta^\alpha(F, a, x) \leq y$ for any δ. Hence

$$x \in [s, t] \Longrightarrow \gamma^\alpha(F, x, b) \leq z \text{ and } \gamma^\alpha(F, a, x) \leq y \tag{9}$$

But as $\gamma^\alpha(F, a, x) + \gamma^\alpha(F, x, b) = \gamma^\alpha(F, a, b) = y + z$, the inequalities in (9) must be equalities. Thus for a given $y, 0 < y < \gamma^\alpha(F, a, b)$, there exists a set $[s, t] \subset [a, b]$ such that $x \in [s, t] \Longrightarrow \gamma^\alpha(F, a, x) = y$ which completes the proof. •

Corollary 13 *If $\gamma^\alpha(F, a, b)$ is finite, $\gamma^\alpha(F, a, x)$ is continuous for $x \in (a, b)$.*

This can be proved using the monotonicity of $\gamma^\alpha(F, a, b)$ in a and b.

Remark: The implication of this result is that no single point has a nonzero mass, or in other words, the mass function is atomless.

The scaling and translation properties of the mass function are similar to those of Hausdorff measure:

Theorem 14 *For $F \subset \mathbf{R}$ and $\lambda \in \mathbf{R}$, let $F + \lambda$ denote the set*

$$F + \lambda = \{x + \lambda : x \in F\}$$

and let λF denote the set

$$\lambda F = \{\lambda x : x \in F\}.$$

Then,

(i) Translation:

$$\gamma^\alpha(F+\lambda, a+\lambda, b+\lambda) = \gamma^\alpha(F, a, b)$$

(ii) Scaling ($\lambda \geq 0$):

$$\gamma^\alpha(\lambda F, \lambda a, \lambda b) = \lambda^\alpha \gamma^\alpha(F, a, b)$$

Remark: If the set F is self-similar so that $\lambda_0 F \cap [\lambda_0 a, \lambda_0 b] = F \cap [\lambda_0 a, \lambda_0 b]$ for a particular λ_0, then the scaling property can be rewritten as

$$\gamma^\alpha(F, \lambda_0 a, \lambda_0 b) = \lambda_0^\alpha \gamma^\alpha(F, a, b).$$

An example is the middle $\frac{1}{3}$ Cantor set C, with $a = 0$, $b = 1$, $\alpha = \ln(2)/\ln(3)$ and $\lambda_0 = \frac{1}{3^n}$ for any positive integer n.

Now we introduce one of the central notions of this paper, viz. the integral staircase function for a set F of the order α. This function, which is a generalization of functions like the Lebesgue-Cantor staircase function, describes how the mass of $F \cap [a, b]$ increases as b increases.

Definition 15 *Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by*

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F, a_0, x) & \text{if } x \geq a_0 \\ -\gamma^\alpha(F, x, a_0) & \text{otherwise.} \end{cases} \tag{10}$$

The number a_0 can be chosen according to convenience. A few properties of $S_F^\alpha(x)$ which are restatements of the corresponding properties of the mass function $\gamma^\alpha(F, a, b)$ are as follows.

Theorem 16 *Let F be a subset of* $\mathbf{R}$, *and let $0 < \alpha \leq 1$. If $\gamma^\alpha(F, a, b)$ is finite, then for all $x, y \in (a, b)$ such that $x < y$, the following statements hold:*

(i) $S_F^\alpha(x)$ is increasing in x.

(ii) If $F \cap (x, y) = \emptyset$, then S_F^α is a constant in $[x, y]$.

(iii) $S_F^\alpha(y) - S_F^\alpha(x) = \gamma^\alpha(F, x, y)$.

(iv) S_F^α is continuous on (a, b).

As an example, we calculate and show the graph of S_C^α for the middle $\frac{1}{3}$ Cantor set C in the section 4, after discussing some results required to calculate it.

3. The γ-dimension

We now consider the sets F for which the mass function $\gamma^\alpha(F, a, b)$ gives the most useful information. Due to the similarity of the definitions of the mass function and the Hausdorff outer measure [1, 3, 4, 5, 6], one might expect that the mass function can be used to define a fractal dimension. It is indeed the case. If $0 < \alpha < \beta \leq 1$,

$$\sigma^\beta[F, P] \leq |P|^{\beta-\alpha} \sigma^\alpha[F, P] \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}$$

so that

$$\gamma_\delta^\beta(F,a,b) \le \delta^{\beta-\alpha}\gamma_\delta^\alpha(F,a,b)\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}$$

Thus in the limit as $\delta \to 0$, we get

$$\gamma^\beta(F,a,b) = 0 \qquad \text{provided } \gamma^\alpha(F,a,b) < \infty \text{ and } \alpha < \beta.$$

It follows that $\gamma^\alpha(F,a,b)$ is infinite upto certain value of α, say α_0, and jumps down to zero $\alpha > \alpha_0$ (if $\alpha_0 < 1$). We call this number the γ-dimension of F. $\gamma^{\alpha_0}(F,a,b)$ itself may be zero, nonzero finite, or infinite. To make the notion of dimension precise,

Definition 17 *The γ-dimension of $F \cap [a,b]$, denoted by* $\dim_\gamma(F\cap[a,b])$, *is*

$$\begin{aligned}\dim_\gamma(F\cap[a,b]) &= \inf\{\alpha : \gamma^\alpha(F,a,b) = 0\}\\ &= \sup\{\alpha : \gamma^\alpha(F,a,b) = \infty\}\end{aligned}$$

4. Comparison with standard definitions

4.1. Comparison with box dimension

Let $\dim_\gamma(F\cap[a,b]) = \alpha$. Then $\gamma^\beta(F,a,b)$ diverges for any $\beta < \alpha$. Thus for any $k > 0$, there exists $\delta_0 > 0$ such that $\delta < \delta_0 \Longrightarrow \gamma_\delta^\beta(F,a,b) > k$.

Let P be any subdivision such that $|P| \le \delta$, and let $N_\delta(F\cap[a,b])$ be the number of nonzero terms in the sum $\sigma^\alpha[F,P]$. Then, for arbitrary but fixed $k > 0$ and $\delta < \delta_0$,

$$k < \gamma_\delta^\beta(F,a,b) \le \frac{N_\delta(F\cap[a,b])\delta^\beta}{\Gamma(\beta+1)}$$

where $0 < \beta < \alpha \le 1$. Thus,

$$\ln(k) \le \ln N_\delta(F\cap[a,b]) + \beta\ln(\delta) - \ln(\Gamma(\beta+1))$$

which implies

$$-\beta\ln(\delta) \le \ln N_\delta(F\cap[a,b]) - \ln(k) - \ln(\Gamma(\beta+1)).$$

Dividing by $-\ln(\delta)$ (which is positive for $\delta < 1$),

$$\beta \le \frac{\ln(N_\delta(F\cap[a,b]))}{-\ln(\delta)} - \frac{\ln(k)-\ln(\Gamma(\beta+1))}{-\ln(\delta)}$$

Taking limit as $\delta \to 0$ and noting that the first term is the definition of the box dimension $\dim_B(F\cap[a,b])$ in the limit and the denominator of the second diverges, we get

$$\beta \le \dim_B(F\cap[a,b]) = \lim_{\delta\to 0}\frac{\ln(N_\delta(F\cap[a,b]))}{-\ln(\delta)}.$$

This is true for any $\beta < \alpha = \dim_\gamma(F\cap[a,b])$, so that

$$\dim_\gamma(F\cap[a,b]) \le \dim_B(F\cap[a,b]),$$

i. e. the γ-dimension of a set is *at most equal* to its box dimension.

An example of a set for which both the dimensions are equal is the middle $\frac{1}{3}$ Cantor set C (Corollary 19 in the next subsection implies that $\dim_\gamma C = \ln 2/\ln 3$).

An example of a set for which these dimensions differ is the set $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. It is known [4] that $\dim_B K = 0.5$, and since K is compact, $\dim_\gamma K = 0$ by corollary 19.

4.2. Comparison with Hausdorff measure and dimension

As the definition of Hausdorff measure involves arbitrary countable covers, it is expected that the Hausdorff dimension be finer than γ-dimension. This is shown to be the case below:

Let $\mathcal{H}^\alpha_\delta(E)$ denote the coarse grained Hausdorff measure of a subset E of $\mathbf{R}$, and $\mathcal{H}^\alpha(E)$ denote the Hausdorff measure [3]. Let P be a subdivision with $|P| \leq \delta$. Those components $[x_i, x_{i+1}]$ of P for which $\theta(F, [x_i, x_{i+1}])$ is nonzero, form a δ-cover of $F \cap [a, b]$. Thus,

$$\sigma^\alpha[F, P] \geq \frac{1}{\Gamma(\alpha+1)} \mathcal{H}^\alpha_\delta(F \cap [a, b]).$$

Since this is true for any P such that $|P| \leq \delta$, it follows that

$$\gamma^\alpha_\delta(F, a, b) \geq \frac{1}{\Gamma(\alpha+1)} \mathcal{H}^\alpha_\delta(F \cap [a, b])$$

for each $\delta > 0$. So taking limit as $\delta \to 0$,

$$\gamma^\alpha(F, a, b) \geq \frac{1}{\Gamma(\alpha+1)} \mathcal{H}^\alpha(F \cap [a, b]). \tag{11}$$

which also implies

$$\dim_{\mathcal{H}}(F \cap [a, b]) \leq \dim_\gamma(F \cap [a, b]),$$

i. e. the γ-dimension of a set is *at least equal* to its Hausdorff dimension.

For compact sets, any open covers, and in particular countable open covers, can be replaced by finite subcovers. So the mass function and Hausdorff measure are expected to be proportional. Theorem 18 below shows that this is indeed the case. Thus for compact sets, the definitions turn out to be equivalent. This theorem signifies that for many sets of practical importance, the mass function and γ-dimension is sufficient and gives intuitively meaningful results.

Theorem 18 *For a compact set $F \subset \mathbf{R}$,*

$$\gamma^\alpha(F, a, b) = \frac{1}{\Gamma(\alpha+1)} \mathcal{H}^\alpha(F \cap [a, b]).$$

Proof: For $\delta > 0$, let $\{A_i, i = 1, 2, \ldots\}$ be any countable cover of $F \cap [a, b]$ such that $\mathrm{diam} A_i \leq \frac{\delta}{2}$ for all i. The sets A_i need not be open or closed. Then

$$\mathcal{H}^\alpha_{\delta/2}(F \cap [a, b]) \leq \sum_i (\mathrm{diam} A_i)^\alpha.$$

Consider closed intervals $B_i = [u_i, v_i]$ where $u_i = \inf A_i$ and $v_i = \sup A_i$. Then $A_i \subset B_i$ and $\mathrm{diam} B_i = \mathrm{diam} A_i$. Thus $\{B_i\}$ forms a cover of $F \cap [a, b]$ and

$$\sum_i (\mathrm{diam} B_i)^\alpha = \sum_i (\mathrm{diam} A_i)^\alpha \geq \mathcal{H}^\alpha_{\delta/2}(F \cap [a, b]).$$

Given $\epsilon \in (0, (\delta/2)^\alpha)$, let $\{C_i\}_{i=1}^\infty$ be the open intervals

$$C_i = \left(u_i - \frac{1}{2} \left(\frac{\epsilon}{2^i} \right)^{\frac{1}{\alpha}},\ v_i + \frac{1}{2} \left(\frac{\epsilon}{2^i} \right)^{\frac{1}{\alpha}} \right).$$

The class $\{C_i\}_{i=1}^{\infty}$ thus forms an open cover of $F \cap [a,b]$ and

$$\text{diam}C_i = \text{diam}A_i + \left(\frac{\epsilon}{2^i}\right)^{\frac{1}{\alpha}} < \delta$$

so that

$$\sum(\text{diam}C_i)^\alpha = \sum\left(\text{diam}A_i + \left(\frac{\epsilon}{2^i}\right)^{\frac{1}{\alpha}}\right)^\alpha.$$

A simple consequence of Jensen's inequality [47], which for the case of two variables assures that $(s_1+s_2)^t \le s_1^t + s_2^t$ for $s_1, s_2 > 0$ and $0 < t < 1$, is that

$$\sum(\text{diam}C_i)^\alpha \le \sum(\text{diam}A_i)^\alpha + \epsilon\sum\frac{1}{2^i} = \sum(\text{diam}A_i)^\alpha + \epsilon. \tag{12}$$

We now show that a finite cover consisting of closed intervals can be constructed. As F is compact, so is $F \cap [a,b]$. Thus a finite subset of $\{C_i\}$ covers $F \cap [a,b]$. We denote this finite subcover by $\{D_i, i = 1, \ldots, n\}$. The D_i are open intervals of the form (a_i, b_i). Without loss of generality we can choose this finite subcover $\{D_i\}$ such that $D_i \not\subset D_j$ whenever $i \neq j$. Further, the sets are labeled such that $a_i \le a_{i+1}$. But as $D_i \not\subset D_{i+1}$ and $D_{i+1} \not\subset D_i$, it implies that $a_i < a_{i+1}$ and $b_i < b_{i+1}$.

Now we consider the closures $\overline{D}_i$ of D_i. As $\{D_i\}$ is a finite subcover out of $\{C_i\}$ and $\{\overline{D}_i\}$ have the same diameters as D_i, it follows from (12) that

$$\sum(\text{diam}\overline{D}_i)^\alpha \le \sum(\text{diam}A_i)^\alpha + \epsilon.$$

Let $I_1 = \overline{D}_1$ and $I_i = \overline{D}_i \backslash D_{i-1}$ for $2 \le i \le n$. The collection $\{I_i\}$ forms a finite cover of $F \cap [a,b]$ by closed intervals, and

$$\sum(\text{diam}I_i)^\alpha \le \sum(\text{diam}A_i)^\alpha + \epsilon.$$

The closed intervals I_i share at the most endpoints. The set of all the endpoints of I_i, $1 \le i \le n$ forms a subdivision P of $[a,b]$ which can be refined to a subdivision Q such that $|Q| \le \delta$ and

$$\Gamma(\alpha+1)\sigma^\alpha[F,Q] = \sum(\text{diam}I_i)^\alpha \le \sum(\text{diam}A_i)^\alpha + \epsilon.$$

Therefore,

$$\Gamma(\alpha+1)\gamma_\delta^\alpha(F,a,b) \le \sum(\text{diam}A_i)^\alpha + \epsilon.$$

Since this relation holds for any countable cover $\{A_i\}$ of $F \cap [a,b]$ such that $\text{diam}A_i \le \delta/2$ and for arbitrary $\epsilon > 0$, it follows that

$$\Gamma(\alpha+1)\gamma_\delta^\alpha(F,a,b) \le \mathcal{H}_{\delta/2}^\alpha(F \cap [a,b]) \le \mathcal{H}^\alpha(F \cap [a,b]).$$

Consequently in the limit as $\delta \to 0$,

$$\Gamma(\alpha+1)\gamma^\alpha(F,a,b) \le \mathcal{H}^\alpha(F \cap [a,b]). \tag{13}$$

Equations (11) and (13) together imply the required equality. •

Corollary 19 *If $F \subset \mathbf{R}$ is compact, then* $\dim_\gamma F = \dim_{\mathcal{H}} F$.

Example: We now discuss an important prototype example of $S_F^\alpha(x)$. Consider the middle 1/3 Cantor set C (hereafter referred to as the Cantor set). This set is compact and has a Hausdorff dimension $\alpha = \log(2)/\log(3)$. Thus by theorem 18, $\mathcal{H}^\alpha(C \cap [a,b]) = \Gamma(\alpha+1)\gamma^\alpha(C,a,b)$ and $\dim_\gamma C = \dim_{\mathcal{H}} C = \alpha = \log(2)/\log(3)$. Using the self-similarity of C and the monotonicity as well as scaling and translation properties of the mass function (theorem 14), we can calculate S_F^α at each point. A graph of $\Gamma(\alpha+1)S_C^\alpha(x)$ i. e. $\Gamma(\alpha+1)\gamma^\alpha(C,0,x)$ against x is shown in figure 1. This is the Lebesgue-Cantor Staircase function.

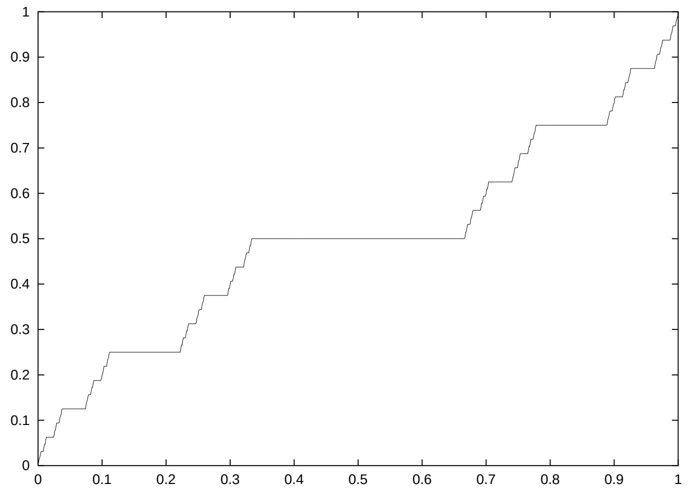

Figure 1. $\Gamma(\alpha+1)S_C^\alpha(x)$: The integral staircase function for the Cantor set

Although the γ-dimension and Hausdorff dimension of a compact set are equal, in general they are significantly different. The difference arises because of the use of subdivisions which are finite, instead of arbitrary countable covers as in Hausdorff measure. In particular, it can be easily seen that $\dim_\gamma \overline{F} = \dim_\gamma F$, where $\overline{F}$ denotes the closure of the set F. This property is comparable to a similar property of box dimension. For example, Consider the set $Q' = Q \cap [0,1]$, the set of all rational numbers in the interval $[0,1]$. For this set, $\dim_\gamma Q' = \dim_B Q' = 1$, whereas $\dim_{\mathcal{H}} Q' = 0$.

As another example, let C_i be the set of the endpoints of the intervals which are formed at the ith state of construction of the middle $\frac{1}{3}$ Cantor set C. Considering $\{[0,\frac{1}{3}],[\frac{2}{3},1]\}$ as the first stage, C_i contains 2^{i+2} points. Let $C' = \cup_{i=1}^\infty C_i$. This set is countable, and dense in C. Therefore,

$$\dim_\gamma C' = \dim_\gamma \overline{C'} = \dim_\gamma C = \frac{\ln(2)}{\ln(3)}$$

whereas

$$\dim_{\mathcal{H}} C' = 0.$$

4.3. Additional remarks

Thus the γ-dimension is finer than the box dimension, but not than Hausdorff dimension. Specifically, the γ-dimension is unaffected by clusters of points unlike the box dimension, as the example of the set $K = \{0, 1, 1/2, 1/3, \ldots\}$ in the previous subsection demonstrates. On the other hand, it is sensitive to countable but dense sets such as rationals.

Summarizing, if $E \subset \mathbf{R}$, then

$$\dim_B E \geq \dim_\gamma E \geq \dim_{\mathcal{H}} E.$$

5. Algorithmic nature of the γ-dimension and the mass function

Since only finite subdivisions are used in the definition of the mass function, it is amenable to algorithmic calculations, and so is the γ-dimension. We present results of one such algorithm here. The algorithm details are given in Appendix D. As in any algorithm to find an approximation to an infimum, this algorithm attempts to find out a subdivision P on a scale δ for which the sum is close to the infimum, i. e. to $\gamma_\delta^\alpha(F, a, b)$. This makes sense in the case of mass function (and not Hausdorff measure) due to the use of finite subdivisions.

The plot in the figure 2 shows (approximations of) $\gamma_{\delta_2}^\alpha(F, a, b)/\gamma_{\delta_1}^\alpha(F, a, b)$, $0 < \delta_2 < \delta_1$, for $\alpha \in (0, 1]$. The value of α for which this graph meets 1 is the possible value of the γ-dimension, since it indicates that γ_δ^α is converging to a nonzero finite value as $\delta \to 0$. It can be plotted for various pairs of scales (δ_1, δ_2) to confirm it.

The first graph is for the middle $\frac{1}{3}$ Cantor set, which meets 1 near the expected value $\ln(2)/\ln(3) = 0.6309\cdots$.

The second graph is for the set $F = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, and it approaches 1 as $\alpha \to 0$, implying that $\dim_\gamma F = 0$. We emphasize that this value differs from the box dimension of this set ($\dim_B F = 0.5$), which is the most commonly used *algorithmic* definition, and the present algorithm is sensitive to this distinction.

6. α-Perfect sets

The correspondence between sets F and their staircase functions S_F^α is many to one. For example, $S_{C'}^\alpha = S_C^\alpha$ where C is the middle $\frac{1}{3}$ Cantor set and $C' = C\backslash\{\frac{2}{3}\}$. Intuitively, it can be said that as the mass function is atomless, removing a single point from C does not change its value.

Another example is a set $D = C \cup E$ where $E \subset (\frac{1}{3}, \frac{2}{3})$ satisfying $\dim_\gamma E < \alpha$, $\alpha = \ln(2)/\ln(3)$. Then it can be seen that $S_D^\alpha(x) = S_C^\alpha(x)$ for all x. Thus adding a lower dimensional set need not change the value of the staircase function either. We call the sets giving rise to the same staircase function as staircasewise congruent:

Definition 20 *Let $F \subset \mathbf{R}$ and $G \subset \mathbf{R}$ be such that $\dim_\gamma F = \dim_\gamma G = \alpha$, $\alpha \in (0, 1]$. Then F and G are said to be staircasewise congruent if $S_F^\alpha(x)$ and $S_G^\alpha(x)$ are finite and*

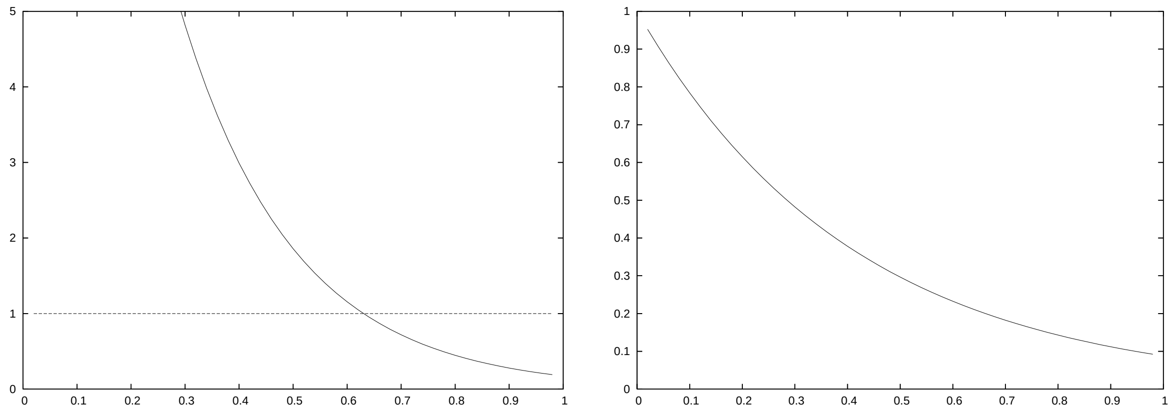

Figure 2. Graph of $\gamma^\alpha_{\delta_2}(F,a,b)/\gamma^\alpha_{\delta_1}(F,a,b)$ against α, with $r_1 = 10^{-2}, r_2 = 10^{-4}$.
Left: for the middle $\frac{1}{3}$ Cantor set
Right: for the set $F = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

equal for all $x \in \mathbf{R}$.

This congruence being an equivalence relation, we denote the equivalence class of sets containing F *by* $\mathcal{E}^\alpha_F$. *Thus if* G *is in* $\mathcal{E}^\alpha_F$, *then* $S^\alpha_G = S^\alpha_F$ *and* $\mathcal{E}^\alpha_G = \mathcal{E}^\alpha_F$.

The above examples intuitively suggest that not all points or subsets contribute to the staircase function. Now we proceed to select a representative set out of the equivalence class which, intuitively speaking, has exactly those points at which S^α_F "changes". To choose only the points where a function "changes", we need the following definition.

Definition 21 *We say that a point* x *is a point of change of a function* f, *if* f *is not constant over any open interval* (c, d) *containing* x. *The set of all points of change of* f *is called the set of change of* f *and is denoted by* $\mathrm{Sch} f$.

Thus $\mathrm{Sch} f_1 = \emptyset$ if $f_1 =$ constant, while $\mathrm{Sch} f_2 = \mathbf{R}$ if $f_2(x) = x$. More importantly we note that if $G \in \mathcal{E}^\alpha_F$, then $S^\alpha_G(x) = S^\alpha_F(x)$ and therefore $\mathrm{Sch}(S^\alpha_G) = \mathrm{Sch}(S^\alpha_F)$. In other words, $\mathrm{Sch}(S^\alpha_F)$ is determined by the equivalence class $\mathcal{E}^\alpha_F$.

The following theorem states that $\mathrm{Sch}(S^\alpha_F)$ itself belongs to $\mathcal{E}^\alpha_F$.

Theorem 22 *Let* $F \subset \mathbf{R}$ *be such that* $S^\alpha_F(x)$ *is finite for all* $x \in \mathbf{R}$ *for* $\alpha = \dim_\gamma F$ *and* $H = \mathrm{Sch}(S^\alpha_F)$. *Then* H *belongs to* $\mathcal{E}^\alpha_F$ *i. e.* $S^\alpha_H = S^\alpha_F$.

Proof: It will be sufficient, in view of definition 15, to prove that $\gamma^\alpha(H, a, b) = \gamma^\alpha(F, a, b)$ for any $a, b \in \mathbf{R}$.

We begin by noting that for $u < v$, if $F \cap [u, v] = \emptyset$, then $\gamma^\alpha(F, u, v) = 0$. Consequently S^α_F is constant on (u, v) implying $(u, v) \cap H = \emptyset$. Then for any $\epsilon' > 0$ such that $u + \epsilon' \leq v - \epsilon'$,

$$\theta(H, [u + \epsilon', v - \epsilon']) = 0. \tag{14}$$

Next, let $\delta > 0$. Then given $\epsilon > 0$, there is a subdivision $P_{[a,b]} = \{y_0, y_1, \ldots, y_n\}$ such that $|P| \leq \delta$ and

$$\sigma^\alpha[F, P] \leq \gamma_\delta^\alpha(F, a, b) + \frac{\epsilon}{2}. \tag{15}$$

If $\theta(F, I) = 1$ for all components I of P, then certainly

$$\sigma^\alpha[H, P] \leq \sigma^\alpha[F, P]. \tag{16}$$

Otherwise, let K be the set of all points of the form

$$c' = c + \left(\frac{\epsilon\,\Gamma(\alpha+1)}{2n}\right)^{\frac{1}{\alpha}}, \ d' = d - \left(\frac{\epsilon\,\Gamma(\alpha+1)}{2n}\right)^{\frac{1}{\alpha}} \tag{17}$$

where c and d are the endpoints of those components I of P such that $\theta(F, I) = 0$. Then $Q_{[a,b]} = P \cup K$ is a refined subdivision and $|Q| \leq |P|$. If $I = [c, d]$ is a component of P such that $\theta(F, I) = 0$, then it contains three components of Q, viz. $[c, c']$, $[c', d']$ and $[d', d]$ where c' and d' are given by (17). The term in $\sigma^\alpha[H, Q]$ corresponding to $[c', d']$ is zero according to (14), and the remaining two contribute at the most $\epsilon/2n$ each. If $\theta(F, I) \neq 0$, then I is also a component of Q and the term corresponding to I in $\sigma^\alpha[H, Q]$ is either zero or is exactly the same as the corresponding term in $\sigma^\alpha[F, P]$. Therefore,

$$\sigma^\alpha[H, Q] \leq \sigma^\alpha[F, P] + \frac{\epsilon}{2}. \tag{18}$$

Thus from (15), (16) and (18), we see that there exists a subdivision Q, such that $|Q| \leq \delta$ and

$$\sigma^\alpha[H, Q] \leq \gamma_\delta^\alpha(F, a, b) + \epsilon.$$

As ϵ is arbitrary, we see that

$$\gamma_\delta^\alpha(H, a, b) \leq \gamma_\delta^\alpha(F, a, b) \tag{19}$$

Now we wish to rule out the possibility that

$$\gamma_\delta^\alpha(H, a, b) < \gamma_\delta^\alpha(F, a, b). \tag{20}$$

Suppose that (20) is true. Then there exists a subdivision $P_1 = \{x_0, \ldots, x_n\}$ such that $|P_1| \leq \delta$ and

$$\sigma^\alpha[H, P_1] < \gamma_\delta^\alpha(F, a, b). \tag{21}$$

From (6) we know that

$$\gamma_\delta^\alpha(F, a, b) \leq \sum_{i=0}^{n-1} \gamma_\delta^\alpha(F, x_i, x_{i+1})$$

so that

$$\sigma^\alpha[H, P_1] < \sum_{i=0}^{n-1} \gamma_\delta^\alpha(F, x_i, x_{i+1}).$$

For this equation to hold, there must be at least one k, $0 \leq k \leq n-1$, such that

$$\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha+1)}\theta(H, [x_k, x_{k+1}]) < \gamma_\delta^\alpha(F, x_k, x_{k+1}) \leq \gamma^\alpha(F, x_k, x_{k+1}).$$

As every quantity is non-negative, it follows that $\gamma^\alpha(F, x_k, x_{k+1}) > 0$, and S_F^α is not constant in $[x_k, x_{k+1}]$. Therefore $\mathrm{Sch}(S_F^\alpha) \cap [x_k, x_{k+1}] \neq \emptyset$ i. e. $H \cap [x_k, x_{k+1}] \neq \emptyset$. So $\theta(H, [x_k, x_{k+1}]) = 1$ and

$$\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha+1)} < \gamma_\delta^\alpha(F, x_k, x_{k+1}).$$

As $Q = \{x_k, x_{k+1}\}$ is a subdivision of $[x_k, x_{k+1}]$ such that $|Q| \leq \delta$,

$$\frac{(x_{k+1} - x_k)^\alpha}{\Gamma(\alpha+1)} = \sigma^\alpha[F, Q] < \gamma_\delta^\alpha(F, x_k, x_{k+1})$$

which is a contradiction by the definition of $\gamma_\delta^\alpha(F, x_k, x_{k+1})$ implying that our assumption (20) is wrong. Thus, (19) is an equality for any $\delta > 0$, and therefore $\gamma^\alpha(H, a, b) = \gamma^\alpha(F, a, b)$. •

Lemma 23 *Let $F \subset \mathbf{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbf{R}$ for $\alpha = \dim_\gamma F$. Then the set $H = \mathrm{Sch}(S_F^\alpha)$ is perfect i. e. H is closed and every point of H is its limit point.*

Proof: Let y be a limit point of H. Then any open interval (c, d) containing y contains a point z of $H = \mathrm{Sch}(S_F^\alpha)$. Therefore S_F^α is not constant on (c, d). Hence $y \in H$ implying that F is closed.

If $x \in H$ is not a limit point of H, then there exists an open interval (c, d) containing x but no other point of H so that $F \cap (c, x) = \emptyset$ and $F \cap (x, d) = \emptyset$. This implies that $S_F^\alpha(d) - S_F^\alpha(c) = (S_F^\alpha(d) - S_F^\alpha(x)) + (S_F^\alpha(x) - S_F^\alpha(c)) = 0$ due to theorem 16(ii). Therefore x is not in $H = \mathrm{Sch}(S_F^\alpha)$ which is a contradiction. •

Now we choose $\mathrm{Sch}(S_F^\alpha)$ as the "canonical" representative of $\mathcal{E}_F^\alpha$:

Definition 24 *Let $F \subset \mathbf{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbf{R}$ for $\alpha = \dim_\gamma F$. Then the set $\mathrm{Sch}(S_F^\alpha)$ is said to be α-perfect, and is said to be the α-perfect representative of $\mathcal{E}_F^\alpha$.*

Thus, every $\mathcal{E}_F^\alpha$ contains a unique α-perfect set. The next theorem states that it is the minimal closed set in $\mathcal{E}_F^\alpha$.

Theorem 25 *An α-perfect set F is the intersection of all the closed sets G in $\mathcal{E}_F^\alpha$. In other words, it is the minimal closed set in $\mathcal{E}_F^\alpha$.*

Proof: Let $\mathcal{G}$ be the class of all closed sets in $\mathcal{E}_F^\alpha$.

As F is perfect, it is closed. Therefore,

$$F \supset \bigcap_{G \in \mathcal{G}} G. \tag{22}$$

Let $G_0 \in \mathcal{G}$ and $x \notin G_0$. Then there is an open interval (c, d) containing x but no point of G_0, as G_0 is closed. This implies that $S_{G_0}^\alpha(c) = S_{G_0}^\alpha(d)$ by theorem 16(ii), and further $S_F^\alpha(c) = S_F^\alpha(d)$ as $G_0 \in \mathcal{E}_F^\alpha$. Since F is α-perfect, we have $F \cap (c, d) = \emptyset$ implying that $x \notin F$. Therefore, $x \in F \Longrightarrow x \in G_0$ for all $G_0 \in \mathcal{G}$, so that

$$F \subset \bigcap_{G \in \mathcal{G}} G. \tag{23}$$

The proof is completed in view of (22) and (23). •

The following lemma is a useful restatement of the definition of an α-perfect set.

Lemma 26 *Let $F \subset \mathbf{R}$ be α-perfect and $x \in F$. If $y < x < z$, then either $S_F^\alpha(y) < S_F^\alpha(x)$ or $S_F^\alpha(x) < S_F^\alpha(z)$ (or both).*

Thus for an α-perfect set F, the lemma assures that if $x \in F$, then the values of $S_F^\alpha(y)$ must be different from $S_F^\alpha(x)$ at all points y on at least one side of x.

As an example, we now show that the middle $\frac{1}{3}$ Cantor set C is α-perfect, for $\alpha = \log(2)/\log(3)$. Let $x \notin C$. As C is closed, there is at least one open interval (c,d) containing x, such that $C \cap (c,d) = \emptyset$. Therefore, S_C^α is constant on (c,d) implying that $x \notin \mathrm{Sch}(S_C^\alpha)$. Hence,

$$\mathrm{Sch}(S_C^\alpha) \subset C. \tag{24}$$

Let x be a point of C. Then x can be represented by $0.x_1x_2x_3\ldots$ where x_i is the ith digit in the ternary representation of x. As $x \in C$, $x_i = 0$ or 2. Let (c,d) be any open interval containing x. Then there is an integer $n > 0$ such that $(x-3^{-n}, x+3^{-n}) \subset (c,d)$. Let D be the set of numbers $y = 0.y_1y_2y_3\ldots$ satisfying

$$y_i = \begin{cases} x_i & 1 \le i \le n+1 \\ 0 \text{ or } 2 & i > n+1 \end{cases}$$

where y_i is the ith digit in the ternary representation of y. Then $D \subset C$ is a scaled down copy of C by a factor 3^{-n-1} and $D \subset (x-3^{-n}, x+3^{-n}) \subset (c,d)$. Thus using the scaling property of γ^α (theorem 14),

$$S_C^\alpha(d) - S_C^\alpha(c) \ge \frac{1}{\Gamma(\alpha+1)}(3^{-n-1})^\alpha = \frac{1}{\Gamma(\alpha+1)}2^{-n-1} > 0$$

where $\alpha = \ln(2)/\ln(3)$. This implies that S_C^α is not constant on (c,d). Thus, $x \in C \Longrightarrow x \in \mathrm{Sch}(S_C^\alpha)$ i. e.

$$C \subset \mathrm{Sch}(S_C^\alpha). \tag{25}$$

From (24) and (25) we see that $C = \mathrm{Sch}(S_C^\alpha)$ implying that C is α-perfect for $\alpha = \log(2)/\log(3)$.

7. F-continuity

In this section we introduce the notation for limit and continuity using topology of $F \subset \mathbf{R}$ with the metric inherited from $\mathbf{R}$. Our purpose in doing so is to distinguish between these notions and ones on $\mathbf{R}$ when they both appear together.

Definition 27 *Let $F \subset \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ and $x \in F$. A number ℓ is said to be the limit of f through the points of F, or simply F-limit of f, as $y \to x$, if given any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$y \in F \text{ and } |y-x| < \delta \Longrightarrow |f(y) - \ell| < \epsilon.$$

If such a number exists, then it is denoted by

$$\ell = F\text{-}\lim_{y\to x} f(y)$$

This definition does not involve values of the function at y if $y \notin F$. Also, F-limit is not defined at points $x \notin F$.

Further we note that since the F-limit as $y \to x$ involves points around x that are in F, it may happen to be a one sided limit at some points (such as the points $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, etc., when $F = C$, the middle $\frac{1}{3}$ Cantor set).

We now introduce the notion of F-continuity which is continuity as far as the values of the function only on the set F are concerned.

Definition 28 *A function $f : \mathbf{R} \to \mathbf{R}$ is said to be F-continuous at $x \in F$ if*

$$f(x) = F\text{-}\lim_{y\to x} f(y)$$

We note that the notion of F-continuity is not defined at $x \notin F$.

It is clear that continuity of $f : \mathbf{R} \to \mathbf{R}$ at $x \in F$ implies F-continuity at x. But the converse is not true. We consider a few examples: Let C be the middle $\frac{1}{3}$ Cantor set. Then the functions $f_1(x) = 1$ and $f_2(x) = x$ are continuous on $[0, 1]$; they are also C-continuous on $C \cap [0, 1]$. In contrast, consider $f_3(x) = \chi_C(x)$ and $f_4(x) = x \cdot \chi_C(x)$ where $\chi_C(x)$ is the characteristic function of C. These functions are C-continuous, but not continuous.

Now we define an analogue of uniform continuity:

Definition 29 *A function $f : \mathbf{R} \to \mathbf{R}$ is said to be uniformly F-continuous on $E \subset F$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$x \in F, y \in E \text{ and } |y - x| < \delta \Longrightarrow |f(y) - f(x)| < \epsilon.$$

It is clear that uniform F-continuity on E implies F-continuity on E. The converse is true only in certain cases:

Theorem 30 *If a function $f : \mathbf{R} \to \mathbf{R}$ is F-continuous on a compact set $E \subset F$, then it is uniformly F-continuous on E.*

8. F^α-Integration

As mentioned in the introduction, we use the Riemann approach for defining integral as it is more direct and algorithmically advantageous. In the definition of F^α-integral below, values of the function only at the points of F are considered. Further, instead of the lengths of subintervals, we consider the difference between the values of the staircase function S_F^α at the endpoints. In this respect, F^α-integral is similar to Riemann-Stiltjes integral [34, 35]. (However, they are in general different, as discussed at the end of this section.)

Definition 31 *The class of functions $f : \mathbf{R} \to \mathbf{R}$ which are bounded on F is denoted by $B(F)$. In other words,*

$$f \in B(F) \Longleftrightarrow -\infty < \inf_{x\in F} f(x) \leq \sup_{x\in F} f(x) < +\infty$$

As the first step, we now define upper and lower sums which approximate the value of the F^{α}-integral.

Definition 32 *Let $f \in B(F)$. Let I be a closed interval. Then,*

$$\begin{aligned} M[f, F, I] &= \sup_{x\in F\cap I} f(x) \qquad && \textit{if } F \cap I \neq \emptyset \\ &= 0 && \textit{otherwise} \end{aligned}$$

and similarly

$$\begin{aligned} m[f, F, I] &= \inf_{x\in F\cap I} f(x) \qquad && \textit{if } F \cap I \neq \emptyset \\ &= 0 && \textit{otherwise} \end{aligned}$$

Definition 33 *Let $S_F^{\alpha}(x)$ be finite for $x \in [a, b]$. Let P be a subdivision of $[a, b]$ with points $x_0, \ldots, x_n$. The upper F^{α}-sum and the lower F^{α}-sum for the function f over the subdivision P are given respectively by*

$$U^{\alpha}[f, F, P] = \sum_{i=0}^{n-1} M[f, F, [x_i, x_{i+1}]](S_F^{\alpha}(x_{i+1}) - S_F^{\alpha}(x_i)) \tag{26}$$

and

$$L^{\alpha}[f, F, P] = \sum_{i=0}^{n-1} m[f, F, [x_i, x_{i+1}]](S_F^{\alpha}(x_{i+1}) - S_F^{\alpha}(x_i)) \tag{27}$$

We emphasize the appearance of intersection $F \cap I$ in the definition of M and m, and also the use of $(S_F^{\alpha}(x_{i+1}) - S_F^{\alpha}(x_i))$ as in a Riemann-Stieltjes sum instead of $(x_{i+1} - x_i)$.

From the definition it is clear that

$$U^{\alpha}[f, F, P] \geq L^{\alpha}[f, F, P]. \tag{28}$$

The following lemma asserts that with refinements, the upper F^{α}-sum decreases and the lower F^{α}-sum increases, both monotonically.

Lemma 34 *Let $F \subset \mathbf{R}$ and $f \in B(F)$. If Q is a refinement of a subdivision P, then $U^{\alpha}[f, F, Q] \leq U^{\alpha}[f, F, P]$ and $L^{\alpha}[f, F, Q] \geq L^{\alpha}[f, F, P]$.*

Proof: To start with, let $P = \{x_0, x_1, \ldots, x_n\}$ and $Q = P \cup \{x'\}$ where $x' \in (x_i, x_{i+1})$. Let $I = [x_i, x_{i+1}]$, $I' = [x_i, x']$, and $I'' = [x', x_{i+1}]$. If there are no points of F in I, then $M[f, F, I] = M[f, F, I'] = M[f, F, I''] = 0$. Otherwise there are two possibilities; either both I' and I'' have points of F, or only one of them, say I' without loss of generality, has points of F.

In the first case,

$$M[f, F, I'] \leq M[f, F, I] \text{ and } M[f, F, I''] \leq M[f, F, I].$$

Thus we have,

$$\begin{aligned}
&M[f,F,I](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) \\
&\qquad = M[f,F,I]\{(S_F^\alpha(x') - S_F^\alpha(x_i)) + (S_F^\alpha(x_{i+1}) - S_F^\alpha(x'))\} \\
&\qquad \geq M[f,F,I'](S_F^\alpha(x') - S_F^\alpha(x_i)) + M[f,F,I''](S_F^\alpha(x_{i+1}) - S_F^\alpha(x')). \qquad (29)
\end{aligned}$$

In the second case, only I' has the points of F. Consequently $S_F^\alpha(x_{i+1}) = S_F^\alpha(x')$ and $M[f,F,I'] = M[f,F,I]$. Thus

$$M[f,F,I](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) = M[f,F,I'](S_F^\alpha(x') - S_F^\alpha(x_i)) \qquad (30)$$

since $M[f,F,I''] = 0$. Combining (29) and (30), we have

$$\begin{aligned}
&M[f,F,I](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)) \\
&\qquad \geq M[f,F,I'](S_F^\alpha(x') - S_F^\alpha(x_i)) + M[f,F,I''](S_F^\alpha(x_{i+1}) - S_F^\alpha(x')).
\end{aligned}$$

Thus

$$U^\alpha[f,F,Q] \leq U^\alpha[f,F,P].$$

This conclusion can easily be extended for any refinement of P.

By a similar argument, we can prove that

$$L^\alpha[f,F,Q] \geq L^\alpha[f,F,P]$$

which completes the proof. •

Lemma 35 *If P and Q are any two subdivisions of $[a,b]$, then*

$$U^\alpha[f,F,P] \geq L^\alpha[f,F,Q]$$

Proof: As $P \cup Q$ is a refinement of both P and Q, it follows from the above lemma and (28) that

$$U^\alpha[f,F,P] \geq U^\alpha[f,F,P\cup Q] \geq L^\alpha[f,F,P\cup Q] \geq L^\alpha[f,F,Q].$$

•

We are now ready to define the F^α-integral.

Definition 36 *Let F be such that S_F^α is finite on $[a,b]$. For $f \in B(F)$, the lower F^α-integral is given by*

$$\underline{\int_a^b} f(x)\, d_F^\alpha x = \sup_{P_{[a,b]}} L^\alpha[f,F,P] \qquad (31)$$

and the upper F^α-integral is given by

$$\overline{\int_a^b} f(x)\, d_F^\alpha x = \inf_{P_{[a,b]}} U^\alpha[f,F,P] \qquad (32)$$

Both the supremum and infimum are taken over all the subdivisions P of $[a,b]$.

The $\mathrm{d}_F^\alpha x$ appearing in (31) and (32) has no separate meaning; it is just the notation.

It is obvious that

$$\underline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x \leq \overline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x \tag{33}$$

Definition 37 *If $f \in B(F)$, we say that f is F^α-integrable on $[a,b]$ if*

$$\underline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x = \overline{\int_a^b} f(x)\,\mathrm{d}_F^\alpha x$$

In that case the F^α-integral of f on $[a,b]$, denoted by $\int_a^b f(x)\,\mathrm{d}_F^\alpha x$ is given by the common value.

For future use we note the following obvious and useful criterion for proving F^α-integrability:

Lemma 38 *Let $f \in B(F)$. Then f is F^α-integrable on $[a,b]$ if and only if, for any $\epsilon > 0$, there exists a subdivision P of $[a,b]$ such that*

$$U^\alpha[f,F,P] < L^\alpha[f,F,P] + \epsilon.$$

Now we state a sufficient condition for F^α-integrability. The sufficient and necessary conditions will be discussed in a companion paper.

Theorem 39 *Let F be such that $F \cap [a,b]$ is compact and S_F^α is finite on $[a,b]$. Let $f \in B(F)$, and $a < b$. If f is F-continuous on $F \cap [a,b]$, then f is F^α-integrable on $[a,b]$.*

Proof: If $S_F^\alpha(a) = S_F^\alpha(b)$, then the F^α-integral is zero, and the result is obvious.

Now consider the case where $S_F^\alpha(a) \neq S_F^\alpha(b)$. The function f is uniformly F-continuous on $F \cap [a,b]$ as $F \cap [a,b]$ is compact. Thus, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$x, y \in F \cap [a,b] \text{ and } |y - x| < \delta \Longrightarrow |f(y) - f(x)| < \frac{\epsilon}{S_F^\alpha(b) - S_F^\alpha(a)}. \tag{34}$$

Let P be a subdivision such that $|P| < \delta$. Then it can be seen that $U^\alpha[f,F,P] < L^\alpha[f,F,P] + \epsilon$ which completes the proof in view of lemma 38. •

The following property of F^α-integral is expected from any fair definition of an integral:

Theorem 40 *Let $a < b$ and f be an F^α-integrable function on $[a,b]$. Let $c \in (a,b)$. Then, f is F^α-integrable on $[a,c]$ and $[c,b]$. Further,*

$$\int_a^b f(x)\,\mathrm{d}_F^\alpha x = \int_a^c f(x)\,\mathrm{d}_F^\alpha x + \int_c^b f(x)\,\mathrm{d}_F^\alpha x \tag{35}$$

This can be proved in a manner analogous to Riemann integral.

The linearity of F^α-integral and a few other properties are easy to prove:

Theorem 41 *(i) If f is F^α-integrable on $[a,b]$, and λ is any real number, then*

$$\int_a^b \lambda f(x)\,\mathrm{d}_F^\alpha x = \lambda \int_a^b f(x)\,\mathrm{d}_F^\alpha x.$$

(ii) If f and g are F^α-integrable functions on $[a,b]$, then

$$\int_a^b (f(x)+g(x))\,\mathrm{d}_F^\alpha x = \int_a^b f(x)\,\mathrm{d}_F^\alpha x + \int_a^b g(x)\,\mathrm{d}_F^\alpha x.$$

Theorem 42 *Let $f : [a,b] \to [a',b']$ be F^α-integrable and let $s : [a',b'] \to \mathbf{R}$ be a continuous function. Then $s \circ f$ is F^α-integrable over $[a,b]$.*

Theorem 43 *Let f, g be F^α-integrable functions over $[a,b]$. Then fg is also F^α-integrable over $[a,b]$.*

Lemma 44 *If f and g are F^α-integrable over $[a,b]$, and $f(x) \geq g(x)$ for all $x \in F \cap [a,b]$, then*

$$\int_a^b f(x)\,\mathrm{d}_F^\alpha x \geq \int_a^b g(x)\,\mathrm{d}_F^\alpha x$$

Definition 45 *If f is F^α-integrable on $[a,b]$, $a < b$, then*

$$\int_b^a f(x)\,\mathrm{d}_F^\alpha x = -\int_a^b f(x)\,\mathrm{d}_F^\alpha x.$$

A particularly simple but important example, as realized in [19], is the F^α-integral of the characteristic function χ_F of the set F:

Lemma 46 *If $\chi_F(x)$ is the characteristic function of $F \subset \mathbf{R}$, then*

$$\int_a^b \chi_F(x)\,\mathrm{d}_F^\alpha x = S_F^\alpha(b) - S_F^\alpha(a)$$

Proof: For a closed interval $I \subset [a,b]$,

$$\begin{aligned} M[\chi_F, F, I] = m[\chi_F, F, I] &= 1 \qquad \text{if } F \cap I \neq \emptyset \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

so that $M[\chi_F, F, I]$ is zero for a closed interval $I = [c,d]$ only when $S_F^\alpha(d) - S_F^\alpha(c) = 0$. Thus

$$U^\alpha[\chi_F, F, P] = L^\alpha[\chi_F, F, P] = S_F^\alpha(b) - S_F^\alpha(a)$$

for any subdivision P of $[a,b]$. •

As a further example, in Appendix A we calculate the C^α-integral of the function $f(x) = x\,\chi_C(x)$ where C is the middle $\frac{1}{3}$ Cantor set, and $\alpha = \ln(2)/\ln(3)$ is its γ-dimension. The integral is given by equations (A.12) and (A.13).

8.1. Difference between F^α-integral and Riemann-Stieltjes integral

In defining the F^α-integral, the quantity $S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)$ has been used in equations 26 and 27. This makes the F^α-integral similar to Riemann-Stieltjes integral in some respects. However we would like to note a couple of differences:

Firstly, quantities in definition 32 use supremum and infimum over $x \in I \cap F$, i. e. only points of I which are in F are considered. On the other hand, the corresponding quantities in Riemann-Stieltjes integral use supremum and infimum over $x \in I$. Due to this difference, for instance, the F^α-integral of the characteric function χ_C of the middle $\frac{1}{3}$ Cantor set C, over the interval $[0,1]$ is nonzero, where the Riemann-Stieltjes integral of the same function with respect to the staircase function S_F^α is zero, since C is nowhere dense. This applies to any fractal F such that $\dim_\gamma F < 1$.

Further, a derivative which is the inverse of the Riemann-Stieltjes integral in the sense of fundamental theorems of calculus is not generally defined unless the "weight" function used is strictly monotonic. For the F^α-integral, even if the function S_F^α has a zero ordinary derivative almost everywhere in the Lebesgue sense, F^α-derivative acts as its inverse in this sense as described in the next section.

9. F^α-Differentiation

Like the first order derivative, the F^α-derivative is a limit of a quotient. But here the limit is F-limit, and the denominator is the difference in the values of the staircase function S_F^α at two points. Moreover, intuitively speaking, F is typically the set of change of the function, and α is typically the γ-dimension of F.

Definition 47 *If F is an α-perfect set then the F^α-derivative of f at x is*

$$(\mathcal{D}_F^\alpha f)(x) = \begin{cases} F\text{-}\lim\limits_{y\to x} \dfrac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & \text{if } x \in F \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

if the limit exists.

If there is no ambiguity, we may use the notation $\mathcal{D}_F^\alpha f(x)$ in place of $(\mathcal{D}_F^\alpha f)(x)$.

We note that lemma 26 tells us that if $x \in F$, then we would find such points y which are arbitrarily close to x at least on one side of x so that the denominator in the definition is not zero and the RHS in (36) makes sense.

However, since the definition involves F-limit, it may happen to be a one-sided derivative in accordance with the remarks in the section 7.

We now state a necessary condition for the above limit to exist.

Theorem 48 *If $(\mathcal{D}_F^\alpha f)(x)$ exists for all x in (a,b), then $f(x)$ is F-continuous in (a,b).*

The proof is straightforward.

The linearity of the F^α-derivative is an immediate consequence of the definition 47. Thus:

Theorem 49 *(i) Let f be a function on $[a,b]$. If $(\mathcal{D}_F^\alpha f)(x)$ exists for all $x \in [a,b]$, then $(\mathcal{D}_F^\alpha \lambda f)(x)$ exists and*

$$(\mathcal{D}_F^\alpha \lambda f)(x) = \lambda(\mathcal{D}_F^\alpha f)(x).$$

(ii) Let f and g be functions on $[a,b]$. If $(\mathcal{D}_F^\alpha f)(x)$ and $(\mathcal{D}_F^\alpha g)(x)$ exist for all $x \in [a,b]$, then $\mathcal{D}_F^\alpha (f+g)(x)$ exists and

$$\mathcal{D}_F^\alpha (f+g)(x) = (\mathcal{D}_F^\alpha f)(x) + (\mathcal{D}_F^\alpha g)(x).$$

Now we calculate the derivative for two elementary functions. The first does not need a proof:

Lemma 50 *The F^α-derivative of a constant function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = k \in \mathbf{R}$ is zero, i. e.*

$$\mathcal{D}_F^\alpha f = 0.$$

This result is to be contrasted with the classical fractional derivative (Riemann-Liouville, and others) of a constant which is not zero in general [37, 38, 39, 40].

Lemma 51 *The derivative of the integral staircase itself is the characteristic function χ_F of F:*

$$(\mathcal{D}_F^\alpha S_F^\alpha)(x) = \chi_F(x).$$

Proof: If $x \notin F$, $(\mathcal{D}_F^\alpha S_F^\alpha)(x) = 0$.

If $x \in F$, then

$$(\mathcal{D}_F^\alpha S_F^\alpha)(x) = F\text{-}\lim_{y \to x} \frac{S_F^\alpha(y) - S_F^\alpha(x)}{S_F^\alpha(y) - S_F^\alpha(x)} = 1$$

•

This lemma together with lemma 46 can be viewed as the special cases of the fundamental theorems of calculus (section 10) involving S_F^α and its derivative χ_F.

An analogue of Rolle's theorem is:

Theorem 52 *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that* $\mathrm{Sch} f \subset F$ *where F is α-perfect, $\mathcal{D}_F^\alpha f(x)$ is defined for all $x \in [a, b]$, and $f(a) = f(b) = 0$. Then there is a point $c \in F \cap [a, b]$ such that $\mathcal{D}_F^\alpha f(c) \geq 0$ and a point $d \in F \cap [a, b]$ such that $\mathcal{D}_F^\alpha f(d) \leq 0$.*

Proof: If f is zero throughout $[a, b]$, then $\mathcal{D}_F^\alpha f(x) = 0$ for all $x \in [a, b]$ and the result follows in this case.

If $f(y) > 0$ for some $y \in (a, b)$, then as f is continuous, there exists an open interval $(c, d) \subset (a, b)$ containing y such that $f(z) > 0$ for any $z \in (c, d)$. Let (c_0, d_0) be largest such interval. Then $f(c_0) = f(d_0) = 0$. The point $c_0 \in \mathrm{Sch} f \subset F$ as f is positive on the right of c_0. So,

$$\mathcal{D}_F^\alpha f(c_0) = F\text{-}\lim_{z \to c_0} \frac{f(z) - f(c_0)}{S_F^\alpha(z) - S_F^\alpha(c_0)} \geq 0.$$

Similarly $d_0 \in F$ and

$$\mathcal{D}_F^\alpha f(d_0) = F\text{-}\lim_{z \to d_0} \frac{f(z) - f(d_0)}{S_F^\alpha(z) - S_F^\alpha(d_0)} \leq 0.$$

with the same considerations. The points c_0 and d_0 can be identified as points c and d in the statement of the theorem.

If there are no points y such that $f(y) > 0$ and neither is the function zero throughout, then we can choose a point y such that $f(y) < 0$ and proceed in a similar manner. •

Remark: The following example shows that the analogue of Rolle's theorem can not be made more strict which would have implied existence of a point $c \in F$ such that $\mathcal{D}_F^\alpha f(c) = 0$. Let C be the middle $\frac{1}{3}$ Cantor set, and let $\alpha = \ln(2)/\ln(3)$. Define

$$\begin{aligned} f(x) &= S_C^\alpha(x) && 0 \leq x \leq 0.5 \\ &= 1/\Gamma(\alpha+1) - S_C^\alpha(x) && 0.5 < x \leq 1. \end{aligned}$$

This function satisfies $f(0) = f(1) = 0$. Further, it is continuous in the interval $[0, 1]$. Its set of change is C. The C^α-derivative is given by

$$\begin{aligned} \mathcal{D}_C^\alpha f(x) &= \chi_C(x) && 0 \leq x \leq 0.5 \\ &= -\chi_C(x) && 0.5 < x \leq 1. \end{aligned}$$

Thus, $x \in C \Longrightarrow \mathcal{D}_C^\alpha f(x) = \pm 1 \neq 0$.

In general, it can be said that the "fragmented nature" of the fractal F does not allow us to make the analogue of Rolle's theorem as strict as its original version.

Now we state the analogue of the law of the mean.

Corollary 53 *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that its set of change is contained in an α-perfect set $F \subset \mathbf{R}$, $\mathcal{D}_F^\alpha f(x)$ exists at all points $x \in [a, b]$ and $S_F^\alpha(b) \neq S_F^\alpha(a)$. Then there exists a point $c \in F$ such that*

$$\mathcal{D}_F^\alpha f(c) \geq \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)}$$

and a point $d \in F$ such that

$$\mathcal{D}_F^\alpha f(d) \leq \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)}$$

Proof: Let

$$g(x) = (f(x) - f(a)) - \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)}(S_F^\alpha(x) - S_F^\alpha(a))$$

so that the difference between f and g is a constant plus a multiple of $S_F^\alpha(x) - S_F^\alpha(a)$. Now $g(a) = g(b) = 0$ so that we can use the last theorem to say that there exists a point $c \in F$ such that $\mathcal{D}_F^\alpha g(c) \geq 0$ and a point $d \in F$ such that $\mathcal{D}_F^\alpha g(d) \leq 0$. This implies that

$$\mathcal{D}_F^\alpha f(c) - \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)} \geq 0$$

for some $c \in F$, and

$$\mathcal{D}_F^\alpha f(d) - \frac{f(b) - f(a)}{S_F^\alpha(b) - S_F^\alpha(a)} \leq 0$$

for some $d \in F$, which lead to the required relations. •

We had seen earlier that the F^α-derivative of a constant $f(x) = k$ is zero. Now we see that these are the only functions whose F^α-derivatives are zero:

Corollary 54 *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that* $\mathrm{Sch}(f) \subset F$ *and $\mathcal{D}_F^\alpha f(x) = 0$ for all $x \in [a, b]$. Then $f(x) = k$ for any $x \in [a, b]$ where k is some real constant.*

Proof: Suppose, if possible, that the function is not a constant. Then there exist y and z, $y < z$, such that $f(y) \neq f(z)$. This implies either $f(y) < f(z)$ or $f(y) > f(z)$.

Case 1. $f(y) < f(z)$. Then there exists a $c \in F \cap (y, z)$ such that

$$\mathcal{D}_F^\alpha f(c) \geq \frac{f(z) - f(y)}{S_F^\alpha(z) - S_F^\alpha(y)} > 0.$$

Case 2. $f(y) > f(z)$. Then there exists a $d \in F \cap (y, z)$ such that

$$\mathcal{D}_F^\alpha f(d) \leq \frac{f(z) - f(y)}{S_F^\alpha(z) - S_F^\alpha(y)} < 0.$$

In both the cases we have found a point where the derivative is not zero which contradicts our assumption. •

Remark: Again due to the "fragmented nature" of the fractal F, the F^α-differentiability of f is not sufficient to guarantee the result. Further, The additional conditions that f be a continuous function and $\mathrm{Sch}(f) \subset F$ are necessary also in the second fundamental theorem 57 which relies on the last corollary, and the integration by parts rule (theorem 58) which depends on theorem 57.

The F^α-derivative satisfies the analogue of Leibniz rule:

Theorem 55 *If the functions $u : \mathbf{R} \to \mathbf{R}$ and $v : \mathbf{R} \to \mathbf{R}$ are F^α-differentiable, then $f(x) = u(x)v(x)$ is F^α-differentiable, and*

$$\mathcal{D}_F^\alpha(uv)(x) = (\mathcal{D}_F^\alpha u)(x)v(x) + u(x)(\mathcal{D}_F^\alpha v)(x). \tag{37}$$

The proof is straightforward.

10. Fundamental theorems of F^α-calculus

This section relates the F^α-integration and F^α-differentiation as "inverse processes" of each other.

The first fundamental theorem says that the F^α-derivative is the inverse of indefinite F^α-integral.

Theorem 56 *Let $F \subset \mathbf{R}$ be an α-perfect set. If $f \in B(F)$ is an F-continuous function on $F \cap [a, b]$ and*

$$g(x) = \int_a^x f(y)\, \mathrm{d}_F^\alpha y$$

for all $x \in [a, b]$, then

$$\mathcal{D}_F^\alpha g(x) = f(x)\chi_F(x).$$

Proof: If $x \notin F$, then $\mathcal{D}_F^\alpha g(x) = 0$ by definition.

For $x \in F$, if there are points in F arbitrarily close to x on both sides of x, then we have to consider both the following cases:

(i) The set $F \cap (x, z)$ is never empty for $z > x$ and

$$g(z) - g(x) = \int_x^z f(y)\, \mathrm{d}_F^\alpha y.$$

(ii) The set $F \cap (z, x)$ is never empty for $z < x$ and

$$g(x) - g(z) = \int_z^x f(y)\, \mathrm{d}_F^\alpha y.$$

Otherwise we have to consider only one of the cases which is applicable. We consider the first one; the second can be treated similarly.

In the first case, $F \cap (x, z)$ is not empty for any $z > x$. Taking the F-limit as $z \to x$, we get

$$\mathcal{D}_F^\alpha g(x) = F\text{-}\lim_{z\to x} \frac{\int_x^z f(y)\, \mathrm{d}_F^\alpha y}{S_F^\alpha(z) - S_F^\alpha(x)}. \tag{38}$$

Now,

$$m[f, F, [x, z]] \int_x^z \chi_F(y)\, \mathrm{d}_F^\alpha y \leq \int_x^z f(y)\, \mathrm{d}_F^\alpha y \leq M[f, F, [x, z]] \int_x^z \chi_F(y)\, \mathrm{d}_F^\alpha y,$$

and

$$\int_x^z \chi_F(y)\, \mathrm{d}_F^\alpha y = (S_F^\alpha(z) - S_F^\alpha(x))$$

so that

$$m[f, F, [x, z]] \leq \frac{\int_x^z f(y)\, \mathrm{d}_F^\alpha y}{S_F^\alpha(z) - S_F^\alpha(x)} \leq M[f, F, [x, z]]. \tag{39}$$

As f is F-continuous,

$$F\text{-}\lim_{z\to x} m[f, F, [x, z]] = F\text{-}\lim_{z\to x} M[f, F, [x, z]] = f(x) \tag{40}$$

From (38), (39) and (40), we get the required result. •

The second fundamental theorem says that the F^α-integral as a function of upper limit is the inverse of F^α-derivative except for an additive constant.

Theorem 57 *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, F^α-differentiable function such that* $\mathrm{Sch}(f)$ *is contained in an α-perfect set F and $h : \mathbf{R} \to \mathbf{R}$ be F-continuous, such that*

$$h(x)\chi_F(x) = \mathcal{D}_F^\alpha f(x).$$

Then

$$\int_a^b h(x)\, \mathrm{d}_F^\alpha x = f(b) - f(a).$$

Proof: If

$$g(x) = \int_a^x h(x)\, \mathrm{d}_F^\alpha x$$

then $\mathcal{D}_F^\alpha g(x) = h(x)\chi_F(x)$ by the last theorem. Therefore $\mathcal{D}_F^\alpha(g - f)(x) = 0$ for all $x \in [a, b]$. Now corollary 54 implies that $g(x) - f(x) = k$, a constant, or $g(x) = f(x) + k$. Thus,

$$f(b) - f(a) = g(b) - g(a) = g(b) = \int_a^b h(x)\, \mathrm{d}_F^\alpha x$$

which proves the theorem. •

The following theorem states that the F^α-integration can be performed by parts, and can be proved by using fundamental theorem 57 and Leibniz rule (theorem 55):

Theorem 58 *Let the functions $u : \mathbf{R} \to \mathbf{R}, v : \mathbf{R} \to \mathbf{R}$ be such that*

(i) u is continuous on $[a, b]$ and $\mathrm{Sch}(u) \subset F$,

(ii) $\mathcal{D}_F^\alpha u$ exists and is F-continuous on $[a, b]$,

(iii) v is F-continuous on $[a, b]$.

Then,

$$\int_a^b u(x)v(x)\,\mathrm{d}_F^\alpha x = \left[u(x)\int_a^x v(x')\,\mathrm{d}_F^\alpha x'\right]_a^b - \int_a^b \mathcal{D}_F^\alpha u(x)\int_a^x v(x')\,\mathrm{d}_F^\alpha x'\,\mathrm{d}_F^\alpha x. \qquad (41)$$

The proof is straightforward and omitted.

In Appendix B, we discuss examples of repeated F^α-derivatives and F^α-integrals. There we also calculate F^α-derivatives and F^α-integrals of powers $(S_F^\alpha(x))^n$.

Now that the analogies between F^α-calculus and ordinary calculus have become clear, we summarise some of them in section Appendix C for a quick reference.

11. Spaces of F^α-differentiable functions

We begin with the definitions of various spaces. In what follows, SAF stands for “set of all functions” and F is assumed to be a bounded, α-perfect set. Further, $a = \inf F$ and $b = \sup F$ unless mentioned otherwise.

Definition 59 *We introduce the following spaces:*

$B(F)$: SAF bounded on F

$C^0(F)$: SAF which are F-continuous (also denoted by $C(F)$)

$C^k(F)$: $k \in \mathbf{N}$: SAF u such that

$$(\mathcal{D}_F^\alpha)^n u \in C^0(F) \text{ for all } n \le k$$

i. e. SAF that have F-continuous F^α-derivatives upto order k.

We note that for $F = [c, d]$, $C^k(F)$ reduces to the usual space $C^k([c, d])$ of k-times continuously (ordinarily-) differentiable functions since in this case the F^α-derivative reduces to the ordinary derivative.

The following quantity, which is adopted for F^α-derivative from the usual norm on $C^k([c, d])$, can be shown to be a norm on $C^k(F)$:

$$||u|| = \sum_{0\le n\le k} \sup_{x\in F} |[(\mathcal{D}_F^\alpha)^n u](x)|, \quad u \in C^k(F). \qquad (42)$$

Further, $C^k(F)$ can be shown to be complete with respect to this norm.

We now introduce a subclass (definition 60 below) $\tilde{C}^k(F)$ of $C^k(F)$ which can be mapped one-one onto $C^k([c, d])$, the class of ordinarily differentiable functions defined on $[c, d] \subset \mathbf{R}$. Due to such a mapping, many results related to $C^k([c, d])$ can be translated to analogous results for $\tilde{C}^k(F)$.

Definition 60 *If a function u satisfies the condition*

$$S_F^\alpha(x) = S_F^\alpha(y) \Longrightarrow u(x) = u(y) \tag{43}$$

it is said to be ***S_F^α-concordant****. The class of all S_F^α-concordant functions u in $C^0(F)$ is denoted by $\tilde{C}^0(F)$. Further, the class of all functions u in $C^k(F)$ such that $0 \le n \le k \Longrightarrow (\mathcal{D}_F^\alpha)^n u \in \tilde{C}^0(F)$ is denoted by $\tilde{C}^k(F)$.*

It can be easily seen that the norm defined by (42) is also a norm on $\tilde{C}^k(F)$, and further $\tilde{C}^k(F)$ is complete with respect to this norm. Now given a function $u \in \tilde{C}^0(F)$ we can uniquely define a function $v \in C^0([c,d])$ where $c = S_F^\alpha(a)$ and $d = S_F^\alpha(b)$ by the equation

$$v(S_F^\alpha(x)) = u(x).$$

This infact defines a one-to-one onto mapping from $\tilde{C}^0(F)$ to $C^0([c,d])$. This implies that $\tilde{C}^0(F)$ *is separable* since $C^0([c,d])$ is separable [48].

11.1. Separability of $\tilde{C}^k(F)$:

Let Y be the Cartesian product of $k+1$ copies of $\tilde{C}^0(F)$:

$$Y = \prod_{i=0}^{k} Y_i, \quad Y_i = \tilde{C}^0(F), \quad 0 \le i \le k.$$

The elements of Y are thus $(k+1)$-tuples, written in the form $\mathbf{u} = (u_0, \ldots, u_k)$. The space Y being a Cartesian product of separable spaces, is separable [48].

Let Y' be the subspace of Y given by

$$Y' = \{\mathbf{u} = (u_0, \ldots, u_k) \in Y \mid u_n = (D_F^\alpha)^n u_0\}.$$

The space Y' is separable since it is a subspace of Y. Further it is isomorphic to $\tilde{C}^k(F)$, implying that the latter is separable.

11.2. F^α-Integrable functions

In order to generalize the spaces $\tilde{C}^k(F)$ to analogues of Sobolev spaces, we need to consider spaces of F^α-integrable functions and their completion.

Consider the set $\mathcal{L}(F)$ of F^α-integrable functions. This is obviously a vector space, with usual operations of addition and scalar multiplication.

It is easy to see that the quantity

$$\mathcal{N}_p(u) = ||u||_p = \left[\int_a^b |u(x)|^p \, d_F^\alpha x\right]^{1/p} \qquad 1 \le p < \infty \tag{44}$$

is well defined. Further, it satisfies the homogeniety property

$$||\lambda\, u||_p = |\lambda| \, ||u||_p, \qquad \lambda \in \mathbf{R}.$$

Now we follow the convention that p and p' are related by

$$\frac{1}{p} + \frac{1}{p'} = 1 \tag{45}$$

and in that case, for $a, b \geq 0$ and $p \in (1, \infty)$, Young's inequality [48] implies that

$$a\, b \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}. \tag{46}$$

Theorem 61 *(Analogue of Hölder's ineqality) For $u, v \in \mathcal{L}(F)$ and $p \in (1, \infty)$,*

$$\int_a^b |f(x)g(x)| \, \mathrm{d}_F^\alpha x \leq \mathcal{N}_p(f)\mathcal{N}_{p'}(g) \tag{47}$$

Proof: If either $\mathcal{N}_p(f)$ or $\mathcal{N}_{p'}(g)$ is zero, the result is obvious. Otherwise, using (46) with

$$a = \frac{|f(x)|}{\mathcal{N}_p(f)} \qquad \text{and} \qquad b = \frac{|g(x)|}{\mathcal{N}_{p'}(g)}$$

we have

$$\frac{|f(x)|}{\mathcal{N}_p(f)} \cdot \frac{|g(x)|}{\mathcal{N}_{p'}(g)} \leq \frac{1}{p}\frac{|f(x)|^p}{\mathcal{N}_p^p(f)} + \frac{1}{p'}\frac{|g(x)|^{p'}}{\mathcal{N}_{p'}^{p'}(g)} \tag{48}$$

for all $x \in F$. F^α-integrating (48) and using (45), we get the required result. •

Theorem 62 *(Analogue of Minskowski's inequality) For $1 \leq p < \infty$ and $f, g \in \mathcal{L}(F)$, we have*

$$\mathcal{N}_p(f+g) \leq \mathcal{N}_p(f) + \mathcal{N}_p(g) \tag{49}$$

Proof: The case $p = 1$ is obvious. For $p > 1$

$$\mathcal{N}_p^p(f+g) \leq \int_a^b |f(x)|\, |f(x)+g(x)|^{p-1} \, \mathrm{d}_F^\alpha x + \int_a^b |g(x)|\, |f(x)+g(x)|^{p-1} \, \mathrm{d}_F^\alpha x.$$

Now using (47),

$$\begin{aligned}
&\int_a^b |f(x)|\, |f(x)+g(x)|^{p-1} \, \mathrm{d}_F^\alpha x \\
&\qquad \leq \mathcal{N}_p(f)\, \mathcal{N}_{p'}(|f(x)+g(x)|^{p-1}) \\
&\qquad = \mathcal{N}_p(f) \left[\int_a^b |f(x)+g(x)|^{(p-1)p'} \, \mathrm{d}_F^\alpha x\right]^{1/p'} \\
&\qquad = \mathcal{N}_p(f) \left[\int_a^b |f(x)+g(x)|^{p} \, \mathrm{d}_F^\alpha x\right]^{(p-1)/p} \\
&\qquad = \mathcal{N}_p(f)\, \mathcal{N}_p^{p-1}(f+g).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_a^b |g(x)|\, |f(x)+g(x)|^{p-1} \, \mathrm{d}_F^\alpha x \\
&\qquad \leq \mathcal{N}_p(g)\, \mathcal{N}_p^{p-1}(f+g).
\end{aligned}$$

Thus,

$$\mathcal{N}_p^p(f+g) \leq \mathcal{N}_p(f)\, \mathcal{N}_p^{p-1}(f+g) + \mathcal{N}_p(g)\, \mathcal{N}_p^{p-1}(f+g)$$

which implies the required result. •

This proves the triangle inequality for $\mathcal{N}_p$. Therefore, $\mathcal{N}_p$ is a semi-norm. Now we identify an appropriate space so that $\mathcal{N}_p$ acts as a norm on it.

Lemma 63 *For two functions $f, g \in \mathcal{L}(F)$, $\mathcal{N}_p(f-g) = 0$ for $p > 1$, if and only if $\mathcal{N}_1(f-g) = 0$.*

The proof is straightforward and omitted.

Definition 64 *We say that two functions $f, g \in \mathcal{L}(F)$ are $\mathcal{N}_p$-equivalent if*

$$\mathcal{N}_p(f-g) = 0.$$

This equivalence relation partitions $\mathcal{L}(F)$ into equivalence classes of functions. Now we define $L'(F)$ to be the space of these equivalence classes.

The space $L'(F)$ is a vector space with addition and scalar multiplication defined appropriately.

The function $||\cdot||_p = \mathcal{N}_p$ acts as a norm on $L'(F)$. When it is desired to talk of $L'(F)$ with a specific norm $\mathcal{N}_p$, the notation $L'_p(F)$ will be used.

It is obvious that $L'_p(F)$ is not complete. But it can be completed using the standard procedure of identifying equivalent Cauchy sequences:

Definition 65 *We say that two Cauchy sequences $\{f_n\}, \{g_n\}$ are $\mathcal{N}_p$-equivalent if*

$$\lim_{n\to\infty} ||f_n - g_n||_p = 0.$$

This equivalence relation partitions the set of sequences in $L'_p(F)$ into equivalence classes. The set of the equivalence classes of sequences in $L'_p(F)$ is denoted by $L_p(F)$.

Thus $L_p(F)$ is complete by definition, and therefore is a Banach space.

Theorem 66 *The spaces $L'_p(F)$ and $L_p(F)$ are separable.*

The proof will be included in a companian paper.

11.3. Analogues of Abstract Sobolev spaces

Let J be a finite set of nonnegative integers $\{j_1, \ldots, j_m\}$ not exceeding a fixed integer $k \geq 0$, and containing 0. Let $\{X_j, ||\cdot||_{X_j}\} = \{X_j\}_{j\in J}$ be a family of Banach spaces X_j with norms $||\cdot||_{X_j}$. Denote by X their Cartesian product:

$$X = \prod_{j\in J} X_j$$

The members of X are tuples written in the form $\mathbf{u} = (u_{j_1}, \ldots, u_{j_m})$. The set X is a vector space with usual addition and scalar multiplication. Further, a norm can be defined on X such that for $\mathbf{u} = (u_{j_1}, \ldots, u_{j_m}) \in X$,

$$||\mathbf{u}||_X = \sum_{j\in J} ||u_j||_{X_j}.$$

Thus, X is a Banach space, and is separable if and only if each of the $X_j, j \in J$ is separable [48].

From now on, we will take $X_j = L_p(F)$ for each $j \in J$, where $p \in [1, \infty)$ is fixed.

We know that $\tilde{C}^\infty(F) \subset L_p(F)$, and if $u \in \tilde{C}^\infty(F)$, then $\mathcal{D}_F^\alpha u \in \tilde{C}^\infty(F) \subset L_p(F)$.

For $u \in \tilde{C}^\infty(F)$, put

$$||u||_J = \sum_{j\in J} ||(\mathcal{D}_F^\alpha)^j u||_p.$$

Then $||\cdot||_J$ acts as a norm on $\tilde{C}^\infty(F)$. The space $\tilde{C}^\infty(F)$ is in general not complete under this norm. A construction analogous to abstract Sobolev spaces will make this space complete.

Define a mapping $I_J : \tilde{C}^\infty(F) \to X$ by the relation

$$I_J(u) = ((\mathcal{D}_F^\alpha)^{j_1} u, \ldots, (\mathcal{D}_F^\alpha)^{j_m} u)$$

Further, define a projection operator $P_n : X \to X_n$, $n \in J$, by

$$P_n(\mathbf{u} = (u_{j_1}, \ldots, u_{j_m})) = u_n.$$

The mapping I_J is linear and isometric, i. e. for $u \in \tilde{C}^\infty(F)$,

$$||I_J(u)||_X = ||u||_J.$$

Denote the image $I_J(\tilde{C}^\infty(F))$ by $[\tilde{Y}^{k,p}(F)]$. Then I_J is an isometric isomorphism between $\tilde{C}^\infty(F)$ and $[\tilde{Y}^{k,p}(F)]$.

The mapping P_n is a continuous linear mapping from X to X_n.

Now let us denote the closure of $[\tilde{Y}^{k,p}(F)]$ in the topology of X by $[\tilde{W}^{k,p}(F)]$.

Since $[\tilde{W}^{k,p}(F)]$ is closed in X, it is a Banach space. Further, since X is separable, so is $[\tilde{W}^{k,p}(F)]$. Now we define an abstract Sobolev space $\tilde{W}^{k,p}(F)$, based on F^α-calculus, to be

$$\tilde{W}^{k,p}(F) = P_0([\tilde{W}^{k,p}(F)]).$$

It is a *Banach space* as is easily seen, and is *separable* since $L_p(F)$ is separable.

Further, the **abstract Sobolev F^α-derivative** of $u \in \tilde{W}^{k,p}(F)$ is defined as

$$(\mathcal{D}_F^\alpha)^j_{\tilde{W}} u = P_j(P_0^{-1}(u)).$$

In the next section we discuss an example F^α-differential equation in which the Sobolev formalism becomes necessary.

12. Examples and applications of F^α-Differential equations

In this section we briefly touch a couple of prototype examples of F^α-differential equations. These equations, as the following examples demonstrate, are suitable for modelilng a class of nonlinear (in particular sublinear) behaviors. One could write such equations for deterministic as well as stochastic dynamical systems. The F^α-differential equations is the main topic of a subsequent work [49] and is discussed here briefly.

F^α-diffusion equation

Firstly we revisit the local fractional diffusion equation proposed in [19] and also discussed partly in [20]. This equation is of the form

$$\mathcal{D}^\alpha_{F,t}W(x,t) = \frac{\chi_F(t)}{2}\frac{\partial^2}{\partial x^2}W(x,t). \tag{50}$$

where the density W is defined as a function of two arguments $(x,t) \in \mathbf{R}\times\mathbf{R}$ and with a slight change of notation $D^\alpha_{F,t}$ denotes the partial F^α-derivative with respect to time t, χ_F being the characteristic function of F. (This equation may be compared with ordinary diffusion equation $\frac{\partial W(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}W(x,t)$.) The Riemann integral like prescription given in [19] had enabled one to construct a new exact solution. This solution is

$$W(x,t) = \frac{1}{(2\pi S^\alpha_F(t))^{\frac{1}{2}}}\exp\left(\frac{-x^2}{2S^\alpha_F(t)}\right), \qquad W(x,0) = \delta(x). \tag{51}$$

This can be recognized as a subdiffusive solution, since S^α_F is known to be bounded by kt^α, k constant, in simple cases including Cantor sets.

Lutz [46] has considered two processes viz. Fractional Brownian motion (FBM) and Fractal-time processes (FTP) associated with anomalous diffusion. While FBM is local in time, the FTP is nonlocal. But both are non-Markovian. On the other hand the process associated with (50) of the present paper involves formulation in terms of local operators. Further the process, when time is restricted to the fractal set F, is Markovian: See [19] where Chapman-Kolmogorov equation is seen to be satisfied.

The solution to FBM discussed by Lutz, describing Gaussian transport, viz.

$$P_{\mathrm{FBM}}(x,t) = \frac{1}{(4\pi D t^\alpha)^{\frac{1}{2}}}\exp\left(\frac{-x^2}{4Dt^\alpha}\right)$$

to the FBM diffusion equation

$$\frac{\partial}{\partial t}P_{\mathrm{FBM}}(x,t) = \alpha D t^{\alpha-1}\frac{\partial^2}{\partial x^2}P_{\mathrm{FBM}}(x,t)$$

actually corresponds to our solution

$$W(x,t) = \frac{1}{(2\pi S^\alpha_F(t))^{\frac{1}{2}}}\exp\left(\frac{-x^2}{2S^\alpha_F(t)}\right)$$

since the staircase function S^α_F satisfies the bounds

$$at^\alpha \le S^\alpha_F(t) \le bt^\alpha.$$

The function $W(x,t)$ can also be viewed as Markovian and causal approximation to P_{FBM}. We further note that local operator equations are suitable to formulate causal behaviour.

An important observation at this stage is that: equations like (50) are examples of fractal-time evolution processes.

Motion in a fractally distributed medium

As a second example, we consider one dimensional motion of a particle undergoing friction. First we recall the equation of motion in a continuous (i. e. nonfractal) medium. If the frictional force is proportional to the velocity, the equation of motion can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k(x)v \tag{52}$$

where $k(x)$, the coefficient of friction, may be dependent on the particle position x.

Equation (52) can be reexpressed by considering velocity v as a function of position x. The equation can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -k(x)v.$$

Identifying $\mathrm{d}x/\mathrm{d}t = v$ and assuming $v \neq 0$, the equation becomes

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -k(x) \tag{53}$$

which is readily solved by integrating $k(x)$ if $k(x)$, which models the frictional medium, is smooth.

If the underlying medium is a fractal, then (53) is inadequate to model the motion. Instead we propose the F^α-differential equation of the form

$$\mathcal{D}_F^\alpha v(x) = -k(x) \tag{54}$$

for this scenario. Here, the set F is the support of $k(x)$ which describes the underlying fractal medium, and α is the γ-dimension of F. (If F is not α-perfect, then the set $\mathrm{Sch}(S_F^\alpha)$ can be chosen instead.) The function $k(x)$ may be called fractional coefficient of friction due to its physical dimensions.

The solution of (54) is easily seen to be

$$v(x) = v_0 - \int_{x_0}^{x} k(x')\,\mathrm{d}_F^\alpha x' \tag{55}$$

where v_0 and x_0 are the initial velocity and position respectively. In a simple case where $k(x)$ is uniform on the fractal i. e. $k(x) = \kappa\chi_F(x)$ where κ is a constant, (55) reduces to

$$v(x) = v_0 - \kappa(S_F^\alpha(x) - S_F^\alpha(x_0)).$$

In the extreme cases we obtain back the classical behaviour: (i) If F is empty (frictionless case), then $v(x) = v_0$; (ii) If $F = \mathbf{R}$ (uniform medium) then $v(x) = v_0 - \kappa(x - x_0)$.

The time dependence of x is given by

$$t(x) = \int_{x_0}^{x} \frac{1}{v(x')}\,\mathrm{d}x'$$

where $t(x)$ is the time required to reach the position x.

A differential equation requiring Sobolev spaces formalism

Suppose F is an α-perfect bounded set, $0 \in F$, and for any $\delta > 0$, both the sets $(-\delta, 0) \cap F$ and $(0, \delta) \cap F$ are nonempty.

Consider the F^α-differential equation:

$$(\mathcal{D}_F^\alpha f)(x) = g(x), \tag{56}$$

where

$$\begin{aligned} g(x) &= \mathrm{sgn}(x)\ \chi_F(x), \\ \mathrm{sgn}(x) &= +1 && \text{if } x > 0, \\ &= -1 && \text{if } x < 0, \\ &= 0 && \text{if } x = 0. \end{aligned}$$

Then,

$$\begin{aligned} f(x) &= S_F^\alpha(x) - S_F^\alpha(0) \quad \text{if } x \geq 0, \\ &= S_F^\alpha(0) - S_F^\alpha(x) \quad \text{if } x < 0 \end{aligned}$$

should be a solution. But f is not F^α-differentiable at $x = 0$. Thus we need to look at the abstract sobolev derivative of f.

Consider the function

$$g_\lambda(x) = \frac{2}{\pi}\, \chi_F(x)\ \tan^{-1}(\lambda S_F^\alpha(x)).$$

The equation (56) with g_λ in place of g has a solution:

$$f_\lambda(x) = \frac{2}{\pi}\, x\ \tan^{-1}(\lambda S_F^\alpha(x)) - \frac{1}{\lambda\pi} \ln(1 + (\lambda S_F^\alpha(x))^2).$$

The function f_λ is F^α-differentiable at all $x \in F$.

Now in a Sobolev space $\tilde{W}^{1,p}$, $1 \leq p < \infty$,

$$\lim_{\lambda\to\infty} g_\lambda(x) = g(x)$$

and

$$\lim_{\lambda\to\infty} f_\lambda(x) = f(x)$$

implying that g is the abstract sobolev derivative of f and f is a solution of (56) in this sense.

13. Concluding remarks

In this paper we have developed a calculus on fractal subsets of the real line. This developement involved the identification of the special role played by staircase functions associated with fractal sets, which may be compared with the role of independent variable itself in ordinary calculus. In particular, F^α-integrals and F^α-derivatives (of order α, $0 < \alpha \leq 1$) are defined using staircase functions for sets F of dimension α.

In contrast with the classical fractional calculus, the notions of F^α-derivatives and F^α-integrals are specifically tailored for fractals of dimension α and thus provide suitable operators on fractals. Further, they reduce to ordinary derivative and Riemann integral respectively, when $F = \mathbf{R}$ and $\alpha = 1$.

Much of the developement of the F^α-calculus is carried in analogy with the ordinary calculus. Several results and techniques of ordinary calculus, including the Leibniz rule, the fundamental theorems of calculus, the technique of integration by parts etc. have analogues in this calculus. Specifically we have adopted Riemann approach for F^α-integrations. This approach is intuitively direct and algorithmically advantageous. Further, it can possibly be generalised using Kurzweil-Henstock integration schemes [50, 51]. Work is in progress in this direction.

In the process of the developement of the F^α-integrals we have introduced α-mass or mass function associated with a fractal subset F of the real line. This lead us to introduce the γ-dimension in section 3. This dimension is finer than the box-dimension. Though it is not as fine as the Hausdorff dimension, it is specific to the developement of calculus here and we expect it to be associated naturally with algorithms and numerical schemes based on the present calculus.

The spaces of F^α-differentiable and F^α-integrable functions are analysed. In particular, Soboleve spaces are constructed on sublinear fractals and the F^α-differentiability is generalized using Sobolev-like construction.

We have also discussed simple models based on F^α-differential equations. The solutions of F^α-differential equations naturally involve staircase-like functions. Staircase functions such as the Lebesgue Cantor staircase function are known to be bounded by sublinear power laws. Also, they "change" or "evolve" only on a fractal set. Thus, this framework may be useful in modelling many cases of sublinear behaviour, fractal time evolution, fields due to fractal charge distributions, etc. The F^α-differentiability may be used to classify singular probability distribution functions.

Continuous-time dynamical systems are associated with ordinary differential equations, and discrete-time dynamical systems are associated with maps/diffeomorphisms. But as realised in [19], the dynamical systems associated with F^α-differential equations would be those evolving on fractal subsets of time-axis. It would also be of great interest to investigate correspondences between ordinary differential equations and F^α-differential equations. These are explored in a companion paper [49].

There are many obvious directions in which considerations of this paper should be extended. Some of them are mentioned above. Other important directions would be extensions to multivariable case, higher dimensions, developement of differential equations, further exploration of associated function spaces, and variational principles to mention a few. Work is in progress in these directions.

Acknowledgments

Abhay Parvate is thankful to the Council of Scientific and Industrial Research (CSIR), India, for financial assistance. The authors wish to thank Dr. H. Bhate for fruitful discussions.

Appendix A. C^α-Integrating $f(x) = x\,\chi_C(x)$

As an example of F^α-integration, here we calculate

$$g(y) = \int_0^y x\,\chi_C(x)\,\mathrm{d}_C^\alpha x = \int_0^y x\,\mathrm{d}_C^\alpha x \tag{A.1}$$

where C is the middle $\frac{1}{3}$ Cantor set, and $\alpha = \ln(2)/\ln(3)$ is its γ-dimension. The function $f(x) = x\,\chi_C(x)$ is C-continuous on $[0,1]$, hence it is C^α-integrable.

The set $P_n = \{x_i = i/n : 0 \le i \le n\}$ is a subdivision of $[0,1]$. For any component $[x_i, x_{i+1}]$ of P_n, $x_i \le m[f, F, [x_i, x_{i+1}]]$ and $x_{i+1} \ge M[f, F, [x_i, x_{i+1}]]$ if $F \cap [x_i, x_{i+1}] \ne \emptyset$. Therefore,

$$\underline{g(1)} = \lim_{n\to\infty} \sum_{i=0}^{n} \left\{ \frac{i}{n} \left[S_C^\alpha\left(\frac{i+1}{n}\right) - S_C^\alpha\left(\frac{i}{n}\right) \right] \right\} \le L^\alpha[f, F, P_n]$$

and

$$\overline{g(1)} = \lim_{n\to\infty} \sum_{i=0}^{n} \left\{ \frac{i+1}{n} \left[S_C^\alpha\left(\frac{i+1}{n}\right) - S_C^\alpha\left(\frac{i}{n}\right) \right] \right\} \ge U^\alpha[f, F, P_n].$$

Further, it can be seen that

$$\lim_{n\to\infty} [\overline{g(1)} - \underline{g(1)}] = 0.$$

Thus, $g(1)$ can be calculated using the limit

$$g(1) = \lim_{n\to\infty} \sum_{i=0}^{n} \left\{ \frac{i}{n} \left[S_C^\alpha\left(\frac{i+1}{n}\right) - S_C^\alpha\left(\frac{i}{n}\right) \right] \right\}. \tag{A.2}$$

Similarly for integers $m > 0$,

$$g\left(\frac{1}{3^m}\right) = \lim_{n\to\infty} \sum_{i=0}^{n} \left\{ \frac{i}{3^m n} \left[S_C^\alpha\left(\frac{i+1}{3^m n}\right) - S_C^\alpha\left(\frac{i}{3^m n}\right) \right] \right\}. \tag{A.3}$$

Using the self-similarity of C and scaling of S_C^α, we see from (A.2) and (A.3) that

$$g\left(\frac{1}{3^m}\right) = \frac{1}{3^{m(1+\alpha)}}\,g(1) = \frac{1}{6^m}\,g(1). \tag{A.4}$$

We make use of the ternary representation of numbers which simplifies many calculations involving the Cantor set. Any number $y \in [0,1]$ can be represented by the series

$$y = \sum_{i=1}^{\infty} \frac{t_i(y)}{3^i} \tag{A.5}$$

where $t_i(y) = 0, 1$ or 2 is the ith ternary digit of y after ternary point. The number y belongs to C if and only if y has a representation of the form (A.5) where $t_i(y) = 0$ or 2 for all i.

An approximation of $y \in [0,1]$ by a finite number of digits is denoted by

$$T_0(y) = 0 \quad \text{and} \quad T_n(y) = \sum_{i=1}^{n} \frac{t_i(y)}{3^i}. \tag{A.6}$$

The sequence $\{T_n(y)\}_{n=0}^{\infty}$ is a monotonically (but not strictly) increasing sequence whose limit is y. Hence we can write

$$g(y) = \sum_{i=1}^{\infty} \int_{T_{i-1}(y)}^{T_i(y)} x\,\chi(x)\,\mathrm{d}_C^\alpha x = \sum_{i=1}^{\infty} I_i(y) \tag{A.7}$$

where

$$I_i(y) = \int_{T_{i-1}(y)}^{T_i(y)} x\,\chi(x)\,\mathrm{d}_C^\alpha x. \tag{A.8}$$

The quantities $I_i(y)$ can be calculated using the self-similarity of C, the scaling and translation properties of S_C^α (theorem 14), and (A.4). Let $y \in [0,1]$ and let n be any integer such that $i < n \Longrightarrow t_i(y) = 0$ or 2. Then $i < n \Longrightarrow T_i(y) \in C$. For calculating $I_n(y)$, there are three cases corresponding to three possible values of $t_n(y)$:

Case $t_n(y) = 0$**:** Here, $T_{n-1}(y) = T_n(y)$ and $I_n(y) = 0$.

Case $t_n(y) = 1$**:** In this case,

$$T_n(y) - T_{n-1}(y) = \frac{1}{3^n} = \sum_{i=n+1}^{\infty} \frac{2}{3^i}$$

so that there is another sequence $\{t_i(T_n(y))\}$ which does not contain the digit 1, hence $T_n(y) \in C$. The set $[T_{n-1}(y), T_n(y)] \cap C$ can be written as

$$\{z : i < n \Longrightarrow t_i(z) = t_i(y);\ t_n(z) = 0;\ i > n \Longrightarrow t_i(z) = 0 \text{ or } 2\}$$

Therefore it is a scaled down version of C by a factor $1/3^n$ and translated by $T_{n-1}(y)$. Hence writing $x = T_{n-1}(y) + (x - T_{n-1}(y))$, we get

$$\begin{aligned}
I_n(y) &= T_{n-1}(y) \int_{T_{n-1}(y)}^{T_n(y)} \chi_C(x)\,\mathrm{d}_C^\alpha x + \int_{T_{n-1}(y)}^{T_n(y)} (x - T_{n-1}(y))\,\chi_C(x)\,\mathrm{d}_C^\alpha x \\
&= T_{n-1}(y) \int_0^{1/3^n} \chi_C(x)\, d_C^\alpha x + \int_0^{1/3^n} x\,\chi_C(x)\,\mathrm{d}_C^\alpha x \\
&= T_{n-1}(y) \frac{1}{\Gamma} \frac{1}{3^{\alpha n}} + \frac{1}{3^{n(1+\alpha)}}\, g(1) \\
&= \frac{T_{n-1}(y)}{\Gamma 2^n} + \frac{g(1)}{6^n}
\end{aligned}$$

where Γ denotes $\Gamma(\alpha+1)$ for convenience. If $y = T_n(y)$ then $y \in C$. But if $y > T_n(y)$, then as $t_n(y) = 1$ and $t_i \neq 0$ for some $i > n$, therefore $y \notin C$. Thus the half open interval $(T_n(y), y]$ does not intersect C implying that $I_k(y) = 0$ for all $k > n$.

Case $t_n(y) = 2$**:** Here, $T_n(y)$ clearly belongs to C. If D is the set

$$D = \{z : i < n \Longrightarrow t_i(z) = t_i(y);\ t_n(z) = 0;\ i > n \Longrightarrow t_i(z) = 0 \text{ or } 2\}$$

then D is a scaled down version of C by a factor $1/3^n$, $D \subset [T_{n-1}(y), T_n(y)]$, and more specifically, $[T_{n-1}(y), T_n(y)] \cap C = D \cup \{T_n(y)\}$. Therefore by arguments similar to the case $t_n(y) = 1$,

$$I_n(y) = \frac{T_{n-1}(y)}{\Gamma 2^n} + \frac{g(1)}{6^n} \tag{A.9}$$

But unlike the case $t_n(y) = 1$, there is a possibility that $C \cap (T_n(y), y]$ is nonempty so that $I_k(y)$ need not be zero for all $k > n$.

Summarizing,

$$I_n(y) = \begin{cases} 0 & \text{if } t_n(y) = 0 \text{ or} \\ & t_i(y) = 1 \text{ for some } i < n \\ \dfrac{T_{n-1}(y)}{\Gamma 2^n} + \dfrac{g(1)}{6^n} & \text{otherwise.} \end{cases} \tag{A.10}$$

This description requires the value of $g(1)$. It can be found out by putting $y = 1$ in (A.10). If $y = 1$, then $t_i(y) = 2$ for all i. Also,

$$T_n(1) = \sum_{i=1}^{n} \frac{2}{3^i} = 1 - 3^{-n}.$$

Therefore,

$$I_n(1) = \frac{1 - 3^{-(n-1)}}{\Gamma 2^n} + \frac{g(1)}{6^n}.$$

Substituting this in (A.10) and solving (A.7) for $g(1)$, we get

$$g(1) = \frac{1}{2\Gamma}. \tag{A.11}$$

Thus,

$$g(y) = \int_0^y x\chi_C(x)\, \mathrm{d}_F^\alpha x = \sum_{n=1}^{\infty} I_n(y) \tag{A.12}$$

where

$$I_n(y) = \begin{cases} 0 & \text{if } t_n(y) = 0 \text{ or} \\ & t_i(y) = 1 \text{ for some } i < n \\ \dfrac{1}{\Gamma(\alpha+1)} \left[\dfrac{T_{n-1}(y)}{2^n} + \dfrac{1}{2 \cdot 6^n} \right] & \text{otherwise} \end{cases} \tag{A.13}$$

and $T_n(y)$ are given by equations (A.5) and (A.6).

Appendix B. Regarding repeated F^α-integration and F^α-derivative

F^α-derivative

Many dynamical systems are modelled by differential equations involving second and higher order derivatives i. e. derivative operator applied repeatedly. The successive operation of the $\mathcal{D}_F^\alpha$ operator is also possible and gives meaningful results. As an example, let us F^α-differentiate the function $g(x) = (S_F^\alpha(x))^2$ twice, where $F \subset R$ is an α-perfect set. By definition of the derivative,

$$x \notin F \Longrightarrow \mathcal{D}_F^\alpha g(x) = 0. \tag{B.1}$$

If $x \in F$, then

$$\begin{aligned}\mathcal{D}_F^\alpha g(x) &= F\text{-}\lim_{y\to x} \frac{(S_F^\alpha(y))^2 - (S_F^\alpha(x))^2}{S_F^\alpha(y) - S_F^\alpha(x)} \\ &= 2\, S_F^\alpha(x).\end{aligned} \tag{B.2}$$

Equations (B.1) and (B.2) can be combined to give

$$\mathcal{D}_F^\alpha g(x) = 2\, S_F^\alpha(x)\, \chi_F(x) \tag{B.3}$$

where χ_F is the characteristic function of F. As a side remark, denoting $(S_F^\alpha(x))^n$ also by $(S_F^\alpha)^n(x)$, it is easy to generalize this to

$$(\mathcal{D}_F^\alpha(S_F^\alpha)^n)(x) = n(S_F^\alpha(x))^{n-1}\chi_F(x) \tag{B.4}$$

for any integer $n > 0$.

Now we take the second F^α-derivative of g. As far as the operator $\mathcal{D}_F^\alpha$ is concerned, the values of the function outside F make no difference because of the F-limit in its definition. Thus,

$$\begin{aligned}(\mathcal{D}_F^\alpha)^2(S_F^\alpha)^2(x) &= \mathcal{D}_F^\alpha(2\, S_F^\alpha\, \chi_F)(x) \\ &= 2\, \mathcal{D}_F^\alpha\, S_F^\alpha(x) \\ &= 2\, \chi_F(x)\end{aligned} \tag{B.5}$$

Where the last step follows from lemma 51 and linearity (theorem 49).

Apart from the γ-dimension of F, the order α also has another significance. This will be clear from the following example. If C is the Cantor set, then it is known [52] that $S_C^\alpha(x)$ is bounded by the power α of x from below and above:

$$ax^\alpha \leq S_C^\alpha(x) \leq bx^\alpha \tag{B.6}$$

where $\alpha = \ln(2)/\ln(3)$ is the γ-dimension of C. Hence the function g defined above is bounded by power 2α of x:

$$ax^{2\alpha} \leq g(x) = (S_C^\alpha(x))^2 \leq bx^{2\alpha} \tag{B.7}$$

so that

$$x \in F \Longrightarrow 2ax^\alpha \leq \mathcal{D}_C^\alpha g(x) \leq 2bx^\alpha \tag{B.8}$$

and

$$x \in F \Longrightarrow (\mathcal{D}_C^\alpha)^2 g(x) = 2. \tag{B.9}$$

This example demonstrates that F^α-differentiation reduces the power of bounds by α.

Table C1. A few analogies between F^α-calculus and ordinary calculus.

Ordinary calculus	F^α-calculus
R	An α-perfect set F
limit	F-limit
Continuity	F-continuity
$\int_0^y x^n \,\mathrm{d}x = \frac{1}{n+1} y^{n+1}$	$\int_0^y (S_F^\alpha(x))^n \,\mathrm{d}_F^\alpha x = \frac{1}{n+1}(S_F^\alpha(y))^{n+1}$
$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n\, x^{n-1}$	$\mathcal{D}_F^\alpha (S_F^\alpha)^n(x) = n\,(S_F^\alpha(x))^{n-1}\,\chi_F(x)$
Leibniz rule	Theorem 55
Fundamental theorems	Theorems 56 and 57
Integration by parts	Theorem 58

F^α-integration

The F^α-integration can also be carried out in succession. Let F be an α-perfect set. It is already shown that F^α-integration of $\chi_F(x)$ is $S_F^\alpha(x)$:

$$\int_a^{x'} \chi_F(x)\,\mathrm{d}_F^\alpha x = S_F^\alpha(x') \tag{B.10}$$

where for simplicity we have taken $S_F^\alpha(a) = 0$ which is consistent with the definition 15 of the staircase function. Now,

$$g_1(x') \equiv \int_a^{x'} S_F^\alpha(x)\,\mathrm{d}_F^\alpha x = \frac{1}{2}(S_F^\alpha(x'))^2$$

where we have used (B.3) and the fundamental theorem 57. Again, it is easy to generalise this to

$$\int_a^{x'} (S_F^\alpha(x))^n\,\mathrm{d}_F^\alpha x = \frac{1}{n+1}(S_F^\alpha(x'))^{n+1} \tag{B.11}$$

using (B.4).

Appendix C. A few analogies between F^α-calculus and ordinary calculus

The F^α-calculus can be thought of as a generalization of ordinary calculus with Riemann approach. Table C1 shows a few analogies between various quantities.

Appendix D. Algorithm to calculate the mass function and the γ-dimension

The algorithm

Let F be a bounded subset of **R**, with $a = \inf F$ and $b = \sup F$. The algorithm proceeds as follows:

(i) **Initial subdivision.** Divide the interval $[a, b]$ in equal intervals of size $r = (b-a)/N$ for some integer $N > 0$. This forms a subdivision $P = \{x_i = a + ir\}, i = 0, \ldots, N$.

(ii) **Merging.** Form a new subdivision P' consisting of x_0, x_N, and only those points x_i, $1 < i < N$ for which one and only one of the intervals of $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ has a nonempty intersection with F. This step is called merging, since in effect consecutive components of P containing points of F are merged to form P'. Relabel the points of P' in increasing order by y_i, $0 \leq i \leq M \leq N$. Now it is guaranteed that alternate components of P' have nonempty intersection with F. This step reduces the sum σ^α in view of Jensen's ineqality [47].

(iii) **Shrinking.** For every component $[y_i, y_{i+1}]$ that has nonempty intersection with F, let $z_i = \inf F \cap [y_i, y_{i+1}]$ and $z_{i+1} = \sup F \cap [y_i, y_{i+1}]$. Let Q be the new subdivision formed by the points $z_i, 0 \leq i \leq M$. In most cases where the dimension is to be calculated computationally, the set F is available as enumeration of points and the inf and sup do not present algorithmic difficulties. Even otherwise, inf and sup can be approximated using methods such as bisection.

(iv) **Removing isolated points.** For components $[z_i, z_{i+1}]$ which contain fewer points of F than $(z_{i+1} - z_i)T/r$ where T is some small threshold value (e.g. 3), we can replace them by components which are arbitrarily small and centered around those points. Thus we can ignore the contribution from such components while calculating $\sigma^\alpha[F, Q]$. This step takes care of isolated points, since T is the minimum number of points in a bin of size r in order that the particular component contributes.

(v) **Analyzing convergence for various values of α.** Obtain two such subdivisions Q_1 and Q_2, for two values of r, viz. r_1 and r_2, $r_2 < r_1$ (i. e. two values N_1, N_2 of N, $N_2 > N_1$). Plot the ratio $\sigma^\alpha[F, Q_2]/\sigma^\alpha[F, Q_1]$ for all $\alpha \in (0, 1]$. The value of α for which this graph meets 1 is the possible value of the γ-dimension, since it indicates that γ_δ^α is converging to a nonzero finite value as $\delta \to 0$ (since r is some indicator of δ—see below). It can be plotted for various scales δ to confirm it.

Relation between r and δ:

With steps (ii) to (iv), an attempt is being made to form a subdivision Q such that $\sigma^\alpha[F, Q]$ is very close to $\gamma_{|Q|}^\alpha(F, a, b)$, while step (i) tries to control $|Q|$. It is clear that due to step (ii) of merging, r is not an upper bound on $|Q|$.

But for all practical purposes, $|Q|$ behaves monotonically with r. In particular, it is easily seen that if $r_2 = r_1/m, m \in \mathbf{N}$, then $|Q_2| \leq |Q_1|$.

The exact relation between r and $|Q|$, and between $\gamma_{|Q|}^\alpha(F, a, b)$ and $\sigma^\alpha[F, Q]$ are the subjects of further investigation, and may lead to improvements in the algorithm.

References

[1] B. B. Mandelbrot, *The fractal geometry of nature* (Freeman and company, 1977)
[2] A. Bunde and S. Havlin (Eds.), *Fractals in Science* (Springer, 1995)
[3] K. Falconer, *The geometry of fractal sets* (Cambridge university press, 1985)
[4] K. Falconer, *Fractal geometry: Mathematical foundations and applications* (John Wiley and Sons, 1990)

[5] K. Falconer, *Techniques in fractal geometry* (John Wiley and Sons, 1997)

[6] G. A. Edgar, *Integral, probability and fractal measures* (Springer-Verlag, New York, 1998)

[7] K. Ito and H. P. McKean Jr., *Diffusion processes and their sample paths* (Springer, Berlin, Heidelberg, New York, 1974)

[8] R. Metzler, W. G. Glöckle and T. F. Nonnenmacher, "Fractional model equation for anomalous diffusion", *Physica A* **211**(1) 13–24 (1994)

[9] R. Metzler, E. Barkai and J. Klafter, "Anomalous diffusion and relaxation close thermal equilibrium: A fractional Fokker-Planck equation approach", *Phys. Rev. Lett.* **83** 3563 (1999)

[10] R. Hilfer and L. Anton, "Fractional master equations and fractal time random walks", *Phys. Rev.* E **51** R848 (1995)

[11] A. Compte, "Stochastic foundations of fractional dynamics", *Phys. Rev.* E **53(4)** 4191 (1996)

[12] G. M. Zaslavsky, "Fractional kinetic equation for Hamiltonian chaos", *Physica* D **76** 110–122 (1994)

[13] G. M. Zaslavsky, "Chaos, Fractional Kinetics, and Anomalous Transport", *Phys. Rep.* **371** 461–580 (2002)

[14] R. Metzler, E. Barkai and J. Klafter, "Anomalous transport in disordered systems under the influence of external fields", *Physica* A **266** 343–350 (1999)

[15] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: A fractional dynamics approach", *Phys. Rep.* **339** 1–77 (2000)

[16] R. Hilfer, "Fractional diffusion based on Riemann-Liouville fractional derivatives", *Jnl. Phys. Chem.* B **104** 3914 (2000)

[17] K. M. Kolwankar and A. D. Gangal, "Fractional differentiability of nowhere differentiable functions and dimensions", *Chaos* **6**, 505 (1996)

[18] K. M. Kolwankar and A. D. Gangal, "Holder exponents of irregular signals and local fractional derivatives", *Pramana* **48**, 49–68 (1997)

[19] K. M. Kolwankar and A. D. Gangal, "Local fractional Fokker-Planck equation", *Phys. Rev. Lett.* **80** 214 (1998)

[20] K. M. Kolwankar and A. D. Gangal, "Local Fractional Calculus: A Calculus for Fractal Space-Time", in *Fractals: Theory and Applications in Engineering*, eds. M. Dekking, J. Levy Vehel *et al* (Springer, 1999), pp. 171–181

[21] F. B. Adda and J. Cresson, "About Non-differentiable Functions", *J. Math. Anal. Appl.* **263** 721–737 (2001)

[22] A. Babakhani and V. Daftardar-Gejji, "On calculus of local fractional derivatives", *J. Math. Anal. Appl.* **270** 66–79 (2002)

[23] A. Rocco and B. J. West, "Fractional Calculus and the Evolution of Fractal Phenomena", *Physica A* **265** 535 (1999)

[24] M. T. Barlow, *Diffusion on fractals, Lecture notes* (Math. Vol. 1690, Springer, 1998)

[25] J. Kigami, *Analysis on Fractals* (Cambridge Univ. Press, 2000)

[26] K. Dalrymple, R. S. Strichartz and J. P. Vinson, "Fractal differential equations on the Sierpinski gasket", *Jnl. Fourier Anal. and Appl.* **5(2)** 205–287 (1999)

[27] R. S. Strichartz, "Taylor approximations on Sierpinski gasket type fractals", *J. Funct. Anal.* **174(1)** 76–127 (2000)

[28] R. S. Strichartz, *Differential Equations of Fractals*, Princeton University Press, New Jersey 08540, 2006

[29] U. Freiberg and M. Zähle, "Harmonic calculus on fractals—A measure geometric approach I", *Potential Anal.* **16** 265–277 (2002)

[30] U. Freiberg and M. Zähle, "Harmonic calculus on fractals—A measure geometric approach II", *Preprint* (2000)

[31] U. Freiberg, "Analytical properties of measure geometric Krein-Feller-perrators on the real line", *Math. Nachr.* **260** 34–47 (2003)

[32] J. U. Löbus, "Generalized Second Order Differential Operators",, *Math. Nachr.* **152** 229–245 (1991)

[33] T. Fujita, "A Fractional Dimension, Self Similarity and a Generalized Diffusion Operator", *Taniguchi symp. PMMP Katata* 83–90 (1985)

[34] D. V. Widder, *Advanced Calculus*, 2nd Ed. (Prentice-Hall India Pvt Ltd, 1974)

[35] G. E. Shilov and B. L. Gurevich, *Integral, measure and derivative: a unified approach* (Prentice Hall, Inc., 1966)

[36] R. R. Goldberg, *Methods of real analysis* (Oxford and IBH Publishing Co. Pvt. Ltd., 1970)

[37] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives—Theory and Applications* (Gordon and Breach Science Publishers, 1993)

[38] R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific Publ. Co., Singapore, 2000)

[39] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations* (John Wiley, New York, 1993)

[40] K. B. Oldham and J. Spanier, *The fractional calculus* (Academic Press, New York, 1974)

[41] T. Bedford and A. M. Fisher, "Analogues of the Lebesgue density theorem for fractal sets of reals and integers", *Proc. London Math. Soc.* **64(3)** 95–124 (1992)

[42] N. Patzschke and M. Zähle, "Fractional Differentiation in the Self-Affine Case I - Random Functions", *Stochastic Proc. Appl.* **43** 165–175 (1992)

[43] N. Patzschke and M. Zähle, "Fractional Differentiation in the Self-Affine Case II - Extremal Processes", *Stochastic Proc. Appl.* **45** 61–72 (1993)

[44] N. Patzschke and M. Zähle, "Fractional Differentiation in the Self-Affine Case III - The Density of the Cantor Set", *Proc. Amer. Math. Soc.* **117(1)** 137–144 (1993)

[45] M. Zähle, "Fractional Differentiation in the Self-Affine Case V - The Local Degree of Differentiability", *Math. Nachr.* **185** 279–306 (1997)

[46] E. Lutz, "Fractional Langevin equation", *Phys. Rev.* E **64** 051106 (2001)

[47] E. F. Beckenbach and R. Bellman, *Inequalities* (Springer-Verlag, 1961) p. 18

[48] A. Kufner, O. John, S. Fucik, *Function Spaces* (Noordhoff International Publishing, Leyden, 1977)

[49] A. Parvate and A. D. Gangal, "Calculus on fractal subsets of real line – II: Conjugacy with ordinary calculus", *In progress*

[50] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron and Henstock* (American Mathematical Society, 1994)

[51] R. G. Bartle, *A modern theory of Integration* (American Mathematical Society, 2001)

[52] E. Hille and J. D. Tamarkin, "Remarks on a known example of a monotone continuous function", *American Mathematics Monthly* **36** 255–264 (1929)